# Dynamical Instability and Statistical behaviour of N-bodySystems.

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## Abstract

In this paper, with particular emphasis on the N-body problem, we focus our attention on the possibility of a synthetic characterization of the qualitative properties of generic dynamical systems. The goal of this line of research is very ambitious, though up to now many of the attempts have been discouraging. Nevertheless we shouldn't forget the theoretical as well practical relevance held by a possible successful attempt. Indeed, if only it would be concevaible to single out a *synthetic* indicator of (in)stability, we will be able to avoid all the consuming computations needed to *empirically discover* the nature of a particular orbit, perhaps sensibly different with respect to another one very *near* to it. As it is known, this search is tightly related to another *basic question*: the foundations of Classical Statistical Mechanics. So, we reconsider some basic issues concerning the (argued) relationships existing between the dynamical behaviour of many degrees of freedom (mdf) Hamiltonian systems and the possibility of a statistical description of their macroscopic features. The analysis is carried out in the framework of the *geometrical description of dynamics* (shortly, *Geometrodynamical approach*, GDA), which has been shown, [Cipriani 1993, Pettini 1993, Di Bari 1996, Cipriani et al. 1996, Di Bari et al. 1997], to be able to shed lights on some otherwise hidden connections between the qualitative behaviour of dynamical systems and the geometric properties of the underlying manifold. We show how the geometric description of the dynamics allows to get meaningful insights on the features of the manifold where the motion take place and on the relationships existing among these and the stability properties of the dynamics, which in turn help to interpret the evolutionary processes leading to some kind of metastable quasi-equilibrium states.

Nonetheless, we point out that a careful investigation is needed in order to single out the sources of instability on the dynamics and also the conditions justifying a statistical description. Some of the claims existing in the literature have here been reinterpreted, and few of them also corrected to a little extent. The conclusions of previous studies receive here a substantial confirmation, reinforced by means of both a deeper analytical examination and numerical simulations suited to the goal. These latter confirm well known outcomes of the pioneering investigations and result in full agreement with the analytical predictions for what concerns the instability of motion in generic classical Hamiltonian systems. They give further support to the belief about the existence of a hierarchy of time-scales accompanying the evolution of the system towards a succession of ever more detailed local (quasi) equilibrium states. These studies confirm the absence of a direct *(i.e. trivial)* relationship between *average curvature* of the manifold (or even frequency of occurrence of negative values) and dynamical instability. Moreover they clearly single out the full responsibility of fluctuations in geometrical quantities in driving a system towards Chaos, whose onset do not show any correlation with the average values of the curvatures, being these almost always positive. These results highlight a strong dependence on the interaction potential of the connections between qualitative dynamical behaviour and geometric features. These latter lead also to intriguing hints on the statistical properties of generic many dimensional hamiltonian systems. It is also presented a simple analytical derivation of the scaling law behaviours, with respect to the global parameters of the system, as the specific energy or the number of degrees of freedom, of the relevant *geometrodynamical* quantities entering the determination of the stability properties. From these estimates, it emerges once more the strong peculiarity of Newtonian gravitational interaction with respect to all the topics here addressed, namely, the absence of any threshold, in energy or number of particles (insofar $N \gg 1$) or whatsoever parameter, distinguishing among different *regimes* of Chaos; the peculiar character of most geometrical quantities entering in the determination of dynamical instability, which is clearly related to the previous point and that it is also enlightening on the physical sources of the statistical features of N-body self gravitating systems, contrasted with those whose potential is *stable* and *tempered*, in particular with respect to the issue of *ergodicity time*. We will also address another *elementary* issue, related to a careful analysis of *real* gravitational N-body systems, which nevertheless provides the opportunity to gain relevant conceptual improvements, able to cope with the peculiarities mentioned above.

# Introduction

The discipline of Deterministic Chaos, in the last decades, has known an expanding phase hardly comparable with other fields of mathematical physics. The increased evidence of the ubiquity of instability has spurred an improved understanding of nonlinear dynamics. However, at present, as usual for quickly expanding areas of research, after the first phase of *spontaneous growth*, the demand for a more systematic settlement, together with a knowledge of the chaotic phenomena going beyond the one of semi-phenomenological nature, is also rising. The signature of Chaos in realistic models of physical systems is indeed often phenomenological in character, as the occurrence of instability is usually recognized *a posteriori*, looking at its consequences on *unpredictability* through the study of the evolution of a generic perturbation (which represents, from a physical viewpoint, the unavoidable uncertainty on the initial conditions) to a given reference trajectory. However, recently there has been a renewed interest in looking for, at least qualitative, synthetic indicators of dynamical instability in realistic models of physical systems, able to provide an *a priori* criterion. As far as we know, the first *explicit* concrete attempt in this direction dates back to Toda's paper, [Toda 1974], whose goal was to find a concise characterization of the sources of *global* instability related to *local* criteria. As it has been shown (see,`e.g.`, [Benettin et al. 1977, Casati 1975]), the relationship argued by Toda was somewhat too *naïve*, and the link between local and global instabilities turns out to be by far more complex than sought.

Nevertheless, mainly in the field of celestial mechanics, where the dynamical systems under study possess few degrees of freedom (*few*, here, means of order ten, or less), a number of attempts to single out synthetic indicators of instability (although qualitative), have been put forward, looking at the intermingled structure of deformed or destroyed *invariant surfaces* in phase space. This effort has led to a deeper understanding of the conditions responsible for the onset of Chaos in dynamical systems of low dimensionality and, in a sense, not too far from integrable ones.

At the other end, very *far* from integrability, within the sphere of Ergodic Theory, with reference to Statistical Mechanics (although not so close to the latter as the very origin and motivations of the former should let to expect), and consequently mainly for mdf systems, some results have been obtained in the case of abstract dynamical systems, [Anosov 1967, Sinai 1991], but they seem to be of little utility to gain some insights on the behaviour of realistic physical models. Whereas, being mainly focused on conceptual issues on the basic postulates of Analytical and Statistical Mechanics, and for this reason relevant to the present work, the last decades knew a deeper reconsideration of the issue about the origin of unpredictability and of the problem related to the approach to equilibrium for generic mdf Hamiltonian systems (see, `e.g.`, [Galgani 1988] and references therein). This *problem* arose after the first *numerical experiments* performed on a computer, when Fermi, Pasta and Ulam, [Fermi et al. 1965], tried to follow numerically the dynamics of a chain of weakly coupled nonlinear oscillators, in the hope of observing the theoretically expected process of equipartition of energy among normal modes. Those *experiments*, because of their *surprising* results, stimulated a lively debate, and supported the believe that the semi-heuristic argumentations of Boltzmann himself and Jeans about the *exponentially long equipartition times* amongst high and low frequency modes possess a very deep physical meaning[1].

Among the attempts towards intrinsic (or synthetic) criteria to single out the presece of Chaos in models of concrete physical systems, besides the Toda's trial already mentioned, we recall the Krylov's Ph.D. thesis, [Krylov 1979], whose intuitions had important influence on the whole line of research this work belongs to.

Nevertheless, the founding arguments of the investigation of the stability of motions by means of a geometrization of the dynamics date back to the paper by Synge, [Synge 1926], which also partially summarizes previous works of Ricci and Levi-Civita. Although completely unaware of the relevance of exponential instability (of course, in 1926!), he there gave a very clear exposition of the power of the *Geometrodynamics* in gaining intrinsic information on the qualitative behaviour of generic holonomic dynamical systems. Other authors, [Eisenhart 1929], since then, added their own contribution to the development of the subject, extending the range of applicability of the method. In the recent years a new deal of interest has been raised, stimulated by the Krylov's work cited above, (unfortunately diffused worldwide more than two decades after its appearance) and from the mess of papers produced by the abstract ergodic theory (see, `e.g.`, [Sinai 1991]). But only very recently the approach has received a self-consistent settlement able to cope with the occurrence of instability in mdf hamiltonian systems (see [Pettini 1993, Cipriani 1993]) and to give a key to understand the links between local properties of the manifold and the asymptotic qualitative features of the dynamics. In a recent paper, [Di Bari et al. 1997], we extended further the *Geometrodynamical approach*, to handle a wider class of dynamical systems, even with peculiar lagrangian structure, embedding the dynamics in an ensemble of manifolds more general than riemannian: the Finsler ones. This framework enable us to cope with dynamical

[1]For a very interesting account on these pioneering and, for a long time, rather disregarded intuitions, see again [Galgani 1988].

systems otherwise not suitable for a geometrization within the riemannian geometry, [Di Bari and Cipriani 1997a], to get more insights on the nature of instability, to locate the relationships with the geometrical structure of the manifold, [Cipriani and Di Bari 1997c], and to set up a formalism with a *built-in* explicit invariance with respect to any reparametrization of the time variable (see also [Di Bari and Cipriani 1997b]). The GDA has proved to give many interesting insights on the analysis of qualitative properties of dynamical systems (DS) of interest in various fields of Statistical Mechanics, as well in Celestial Mechanics and Cosmology; we will address here a problem that, also looked from a *number criterion*, seats just in the middle between Analytical ($\log N \sim 1$) and Statistical ($\log N \sim 23$) Mechanics. The main concern of the present paper is indeed towards a clarification of the links between geometry and dynamics on one side, and their implications on the statistical behaviour of N-body systems, when $6 \lesssim \log N \lesssim 13$, on the other. In the sequel, we will focus on the properties of the gravitational N-body system and, using the Fermi-Pasta-Ulam (FPU) chain as a *normal reference system*, belonging to the class of mdf systems for which a Statistical Mechanical description is well founded, compares their dynamical, geometrical and statistical features, enlightening from a perhaps nouvel perspective the physical and mathematical peculiarities of Gravity, showing also how to treat successfully some of them.

## Instability and Statistical description.

The relevance of dynamical instability and ergodicity[2] to Statistical Mechanics (SM) has been sometimes questioned, [Farquhar 1964], but it is by now generally accepted that the unpredictability implied by Chaos is a natural ingredient to justify the statistical description of mdf dynamical systems. As stressed in the introduction, however, there is no rigorous, or even convincing, prove of any direct relationship between the instability times, as can be derived from dynamics (Lyapunov exponents, Kolomogorov-Sinai entropy,...) and the time-scales related to a Statistical Mechanical treatment, i.e., those belonging under the name of *relaxation times*. Nevertheless some qualitative relationships exist among them for some interesting and historical mdf systems, as the FPU chain, for which, although differing each other by orders of magnitude, the Lyapunov and the SM relaxation times display the same transition between two different regimes of *weak* and *strong* chaoticity, [Pettini and Cerruti-Sola 1991]. Recently (see [Morbidelli and Froeschlé 1996] and references therein), also for dynamical systems with few degrees of freedom and of particular interest in Celestial Mechanics, some debate raised about such a relation among Lyapunov exponents and the time scale of *diffusion* in phase space.
A thorough treatment of the multiple connection between dynamical instability and Statistical Mechanics is out of place here, [Cipriani 1993, Cipriani and Di Bari 1997b]. Instead, we would like to stress that the suggestions of Boltzmann and Jeans, reminded in the previous section, received, in the framework of Analytical Dynamics, a strong support from the Nekoroshev theorem on the exponentially long time scales of *quasi-conservation* of actions for generic non integrable systems, [Benettin 1988]. This is relevant to the present discussion in giving an explanation to the existence of different regimes of stochasticity in generic Hamiltonian systems of interest in Solid State Physics as well in Astrophysics, and also in demonstrating once more the peculiarity of Newtonian Gravity. Indeed, for the N-body system governed by gravitational interaction, we will support the claim according to which simply do not exist a regime of *quasi integrability*, as this system will be *chaotic* for almost all generic initial conditions[3]. The absence of any integrable limit, makes it hard to find out any threshold, $\varepsilon_*$, for the magnitude of the perturbation, $\varepsilon$, below which the actions are *quasi conserved* for time intervals exponentially long in some power of $(\varepsilon_*/\varepsilon)$; simply, we don't know the state for which $\varepsilon = 0$!

Behind the long standing problem of the justification of the *basic assumptions* at the ground of the Statistical description, [Ma 1988], for generic dynamical systems, lies a slowly evolving set of ideas, all referred to as the *Ergodic hypotesis*[4]. The GDA, applied to N-body gravitational system, discloses evidence of the differences amongst some of the interpretations given of it in order to justify the approach to equilibrium, and, at the same time, reinforces the claim that the processes leading a DS to the hierarchy of intermediate partial equilibria, proceed on time scales whose

[2]We do not intend here neither to confuse nor to assimilate the two properties. We would only point out how even some very basic questions concerning the links between Analytical and Statistical Mechanics are still awaiting for an universally accepted, although not rigorously proved, answer.

[3]We are supposing obviously that $N \gg 1$, and we speak about *chaos* in a not too strict sense, as the long range behaviour of the interaction, more than its short range divergencies, causes the phase and configuration spaces of the system to be non compact.

[4]The *ensemble*, often very inhomogeneus, of concepts indicated under this name has sometimes been presented in a somewhat misleading fashion, leading, from time to time, either to evidently unreliable assumptions or to trivial assertions. This has been an undeserved treatment to Boltzmann efforts to find a strict formulation for a very deep and notwithstanding *ineffable* physical intuition. A thorough and enlightening conceptual re-examination of the issues related to the Boltzmann ideas on the importance of a *physically reasonable formulation* of the Ergodic hypotesis to justify the use of Statistical approach in Mechanics has been carried out recently in a series of paper by G.Gallavotti; for an exhaustive account, see the monography [Gallavotti 1995].

spectrum is *written* in the interaction potential governing its dynamics, and can differ by orders of magnitude passing from a DS to another. We refer to [Cipriani 1993, Cipriani and Di Bari 1997b] for a less synthetic discussion of these issues, and concentrate here mostly on the dynamics of N-body systems and on the insights on their qualitative properties that can be obtained in the framework of GDA.

## Geometrodynamics of evolutionary processes.

The problem of the approach to equilibrium in physical systems with a large number of degrees of freedom is, essentially, a problem of time-scales. The fundamental question is: what time-scale is required such that the system finds itself in a state in which some or all memory of the initial one is lost? A theory of non-equilibrium processes that should be able to give quantitative answers to this question would also provide the hints to solve most of the open issues referring to the connection between Analytical and Statistical Mechanics. This problem has been investigated since the very birth of SM with different methods; here we will exploit the Geometrical approach.
In the following sections we recall briefly the method and then introduce the main Geometrodynamical Indicators (GDI's), whose behaviour determine the qualitative (global) evolution of large N-body systems.
In order to highlight the singular features of Gravity, we apply the method to two potentials, representative of the classes of short and long range interactions respectively. The evaluation of the characteristic time-scales for them is firstly derived on the basis of analytical estimates of the quantities entering the GDI's. On these grounds some of the existing results are reviewed and reinterpreted, giving an (hopefully) coherent scenario. After that, the reliability of the claims based on a trivial extension of methods and concepts borrowed from Ergodic Theory are critically examined and discussed. Still within a semi-analytical approach, we show why that extension results, at least, too much naïve. Using numerical simulations suitably tailored to the goal, we then show that the analytical estimates correctly describe the behaviour of GDI's, and so that the mechanisms driving to Chaos in mdf Hamiltonian systems differ completely from those occurring within the Ergodic theory of abstract DS's.
We conclude comparing the behaviour of the GDI's for the *mathematical* Newtonian potential, on a side, with that of the corresponding quantities related to the *physical* gravitational N-body problem and to the FPU chain, on the other. This allows us to shed some lights on the way a stellar systems could attain its apparent equilibrium state, in spite of the celebrated theorems of *rigorous* Statistical Mechanics on non-existence of a *maximum entropy state* for unscreened Coulombic interactions, which do not possess the property of *stability*, [Ruelle 1988]. The search for the conditions needed in order to guarantee the equivalence between *time* and *phase* averages of geometric quantities, leads also to the conclusion that the peculiarities of Newtonian gravity, reflects themselves in a singularity on the *literal ergodicity time*, singularity which disappears when a realistic interaction, that nevertheless do not modify the very nature of Gravity, is considered, irrespective to the details of the *care*.
Though there are several *geometrizations* of Dynamics, in the following, we exploit mostly that based on the so-called *Jacobi* metric, which results from a straightforward application of the *least action principle* in the form given by Maupertuis, see, e.g., [Goldstein 1980, Arnold 1980].

### The Jacobi Geometrodynamics.

In the framework of a definitely non-perturbative treatment of the Hamiltonian Chaos, the differential geometrical approach stands in the pathway that, started by Krylov with the aim to comprehend the relaxation processes in realistic physical systems, has grown very long in the applications to abstract dynamical systems, acquiring mathematical strictness but only recently broadening the physical interest. The Geometrodynamical approach to a Hamiltonian system reduces it to a geodesic flow over a manifold, M, on which is defined a suitable metric. In the case of the Jacobi geometrization, the manifold is a riemannian one, equipped with the (conformally flat) metric, $\mathsf{g} = \{g_{ab}\}$, whose line element is

$$ds_J^2 \stackrel{\mathrm{def}}{=} g_{ab}dq^a dq^b = [E - \mathcal{U}(\mathbf{q})]\,\eta_{ab}dq^a dq^b = 2\,\mathcal{W}^2\,dt^2, \qquad a,b = 1,\ldots,\mathcal{N}\ ; \tag{1}$$

where $E$ is the total conserved energy, $\mathcal{U}(\mathbf{q})$ is the potential energy, depending on the coordinates $\mathbf{q} = \{q^a\}$ on the *configuration manifold* of the system (with $\mathcal{N}$ degrees of freedom), $\mathcal{W}(\mathbf{q}) \stackrel{\mathrm{def}}{=} [E - \mathcal{U}(\mathbf{q})]$ is the *conformal factor*, whose magnitude numerically equals given the kinetic energy $\mathcal{T}$, and $t$ is the newtonian time. The metric of the $\mathcal{N}$-dimensional *physical space* of the system is thus $\eta_{ab}$, i.e., $2\mathcal{T} = \eta_{ab}\dot{q}^a\dot{q}^b$, which reduces to $\eta_{ab} \equiv \delta_{ab}$ when the space is euclidean and cartesian coordinates are employed; as usual, we denote with a dot differentiation with respect to time. Since we are considering N-body Hamiltonian systems, we have $\mathcal{N} \equiv \mathrm{N}\cdot d$, where $d$ is the dimensionality of the

system. So, for the self-gravitating system we have $d = 3$, whereas $d = 1$ for the FPU chain.
In the language of GDA, the $\mathcal{N}$ second order Lagrangian equations of motion (or the $2\mathcal{N}$ first order Hamiltonian ones) are replaced by the $\mathcal{N}$ geodesics equations

$$\frac{\nabla u^a}{ds} \equiv \frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0 \ , \qquad (a = 1, \dots, \mathcal{N}) \tag{2}$$

where $u^a = dq^a/ds$, $\nabla/ds$ is the total (covariant) derivative along the flow, the $\Gamma^a_{bc}$ are the Christoffel symbols of the metric $\mathsf{g}$ and the summation convention is understood.
For the metric defined in (1), namely $g_{ab} = \mathcal{W}(\mathbf{q})\eta_{ab}$, and using cartesian coordinates, the geodesic equations read[5]:

$$\frac{d^2 q^a}{ds^2} + \frac{1}{2\mathcal{W}}\left[2\frac{\partial \mathcal{W}}{\partial q^c}\frac{dq^c}{ds}\frac{dq^a}{ds} - g^{ac}\frac{\partial \mathcal{W}}{\partial q^c}\right] = 0, \ , \qquad (a = 1, \dots, \mathcal{N}) \tag{3}$$

and give the trajectories on the configuration manifold in terms of the affine parameter $s$ (conciding with the *Maupertuis action*). The *newtonian time* law of percurrence of the trajectory is obtained via the relation $ds = \sqrt{2}\,\mathcal{W}(\mathbf{q})\,dt$, exploiting which one obtain the familiar equations of motion, $\ddot{q}^a + \mathcal{U}^a = 0$.
Once rephrased the dynamics as a geodesic flow, we are left with the determination of the question of stability of geodesic paths on the Jacobi manifold. Within the framework of Hamiltonian dynamics this issue is addressed using the tangent dynamics equations, which determine the evolution of a small deviation vector, $\mathbf{D} = (\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$, in phase-space, describing a perturbation to a given trajectory.
Although the equations for the trajectories coincide with geodesics ones once rephrased in terms of the same parameter, the equations for the *displacements*, which are those relevant to the issue of stability of motion, differ in general. We refer to [Di Bari and Cipriani 1997b] for a discussion of the problem of *equivalence* among different *linearized* equations for displacements and here simply remind that all geodesics have, by definition, unit velocity vector, so, in a sense, the deviation between geodesics gives the distance between *paths* and not between points on them. Another point to be remarked concerns the apparent self consistency of the geometrical description, which allow for a built-in *distance*, lacking in the usual definition for the norm of a vector in phase space, where an Euclidean structure is imposed by hand.

## Geometric description of instability.

As it is well known, in the framework of Riemannian geometry[6] the evolution of a perturbation, $\delta\mathbf{q}$, to a geodesic is described by the Jacobi–Levi-Civita (JLC) equation for *geodesic spread*:

$$\frac{\nabla}{ds}\left(\frac{\nabla \delta q^a}{ds}\right) + \mathcal{H}^a{}_c \delta q^c = 0, \tag{4}$$

where it has been defined the *stability tensor* $\mathcal{H}^a{}_c$, defined along any geodesic and related to the Riemann curvature tensor, $R^a{}_{bcd}$, associated to $g_{ab}$ by

$$\mathcal{H}^a{}_c \stackrel{\text{def}}{=} R^a{}_{bcd} u^b u^d \ , \tag{5}$$

where $\mathbf{u}$ is the unit tangent vector to the geodesic. In the case of Jacobi metric $\boldsymbol{\mathcal{H}}$ reads

$$\begin{aligned} \mathcal{H}^a{}_c \;=\; & \left[\frac{1}{4\mathcal{W}^2}\left(\frac{d\mathcal{W}}{ds}\right)^2 - \frac{1}{2\mathcal{W}}\frac{d^2\mathcal{W}}{ds^2}\right]\delta^a_c + \frac{(\operatorname{grad}\mathcal{W})^2}{4\mathcal{W}^3} u^a u_c + \\ & + \frac{1}{2W}\left[\mathcal{W}_{,bc}\left(u^a u^b - g^{ab}\right) + g^{ab}\mathcal{W}_{,bd}\, u^d u_c\right] + \\ & - \frac{3}{4\mathcal{W}^2}\left[\left(\frac{d\mathcal{W}}{ds}\right)\left(u^a \mathcal{W}_{,c} + g^{ab}\mathcal{W}_{,b}\, u_c\right) - g^{ab}\mathcal{W}_{,b}\,\mathcal{W}_{,c}\right] \end{aligned} \tag{6}$$

which clearly depends also on the position, $\mathbf{q} \in \mathsf{M}$. Here, as usual, $\mathcal{W}_{,a} = \partial\mathcal{W}/\partial q^a$, the summation convention is adopted, and the Euclidean $\mathcal{N}$-dimensional gradient operator acting on a function $f$ defined on $\mathsf{M}$ has been indicated explicitly as $[\operatorname{grad} f]$, rather than $\nabla f$, to avoid confusion with the already defined total derivative along the flow $\nabla/ds$.

[5]Since we restrict to consider mostly Jacobi geometrization, we will from now on omit the subscript in $ds_J$ whenever there is no risk of ambiguity, writing simply $ds$.
[6]And also in more general differential manifolds, see [Di Bari 1996, Di Bari et al. 1997].

Equation (4) contains all the information about the evolution of a congruence of geodesics emanating within an initial distance $z = \|\delta\mathbf{q}\| = (g_{ab}\ \delta q^a \delta q^b)^{1/2}$ from the reference one. The problem for a system with mdf is that the Riemann tensor contains $\mathcal{O}(\mathcal{N}^4)$ components, and the evaluation of the stability tensor $\boldsymbol{\mathcal{H}}$ is an huge task. Nevertheless, just when the dimensionality of the ambient space is high, some assumptions can be made on its *global* properties, averaging in a suitable way. This procedure has been adopted in the past (see, e.g., [Pettini 1993, Cipriani 1993] and references therein) with success in describing the behaviour of mdf systems of interest in Statistical Mechanics. Indeed, if we are interested in the stability properties of the flow, the relevant quantity is the magnitude of the perturbation, measured within the GDA by its norm. Indeed, if $\delta\mathbf{q} = z\boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is a vector of the unitary tangent space at $\mathbf{q}$, $\mathsf{T}_{\mathbf{q}}\mathsf{M}$, the norm $z$ evolves according to:

$$\frac{d^2 z}{ds^2} = \left( -\mathcal{H}_{ac}\nu^a\nu^c + \left\| \frac{\nabla\boldsymbol{\nu}}{ds} \right\|^2 \right) z \tag{7}$$

which is still an *exact* equation derived directly from eq.(4), [Cipriani 1993]. However, eq.(7) contains still the full stability tensor, and the task has so been made easier only partially, reducing to a single one the number of equations to be integrated. To *close* this equation some assumption is needed.

## Geometric Indicators of Instability.

We refer to [Cipriani 1993, Pettini 1993] and subsequent works for the description of the details of the averaging procedure, and we want here recall only some critical points which deserve more cautious treatment than the one given sometimes, [Gurzadyan and Savvidy 1986, Kandrup 1990]. Indeed, one intuitive approximation that help to partially get rid of the huge amount of computational task in the evaluation of the stability tensor $\boldsymbol{\mathcal{H}}$ is suggested by the physical reasoning that we are interested in the evolution of a *generic* small perturbation, however oriented with respect to the *given geodesic*. Actually, the $\boldsymbol{\mathcal{H}}$ tensor contains all the informations about the *local* evolution of *any* perturbation to the system, *local* in the sense that it describes the behaviour of the vectors of tangent space in a neighbourhood of the *state* $(\mathbf{q},\mathbf{u})$. So, a first averaged equation can be derived from eq.(7) letting $\delta\mathbf{q}$ to have a random orientation *with respect to the given flow*. So, as we have by definition that $\|\boldsymbol{\nu}\|^2 \equiv g_{ab}\nu^a\nu^b = 1$, being the Jacobi metric conformally flat, this equation would be consistent with a randomly chosen orientation for $\boldsymbol{\nu}$ if we put $\langle\nu^a\nu^b\rangle = g^{ab}/\mathcal{N}$. Nevertheless, while practically irrelevant when $N \gg 1$, we should make here a comment about the correct way of averaging the deviation over directions. Obviously, given a geodesic, there are $(\mathcal{N}-1)$ independent orientations orthogonal to $\mathbf{u}$, and we can imagine also a perturbation with a non vanishing component along the geodesic itself. This straight argumentation leads to the $\mathcal{N}^{-1}$ factor in the average above. Yet, it follows directly from the very definition of the *stability tensor*, $\boldsymbol{\mathcal{H}}$, i.e., from the simmetry properties of Riemann tensor, that at least one amongst its local eigenvalues vanishes, being that associated with the direction along the flow:

$$\mathcal{H}^a{}_c\ l^c \equiv 0, \qquad \forall\ \mathbf{l} = P\mathbf{u}, \tag{8}$$

with $P(s)$ any scalar quantity. So, a deviation along the geodesic cannot evolve more than linearly in the $s$-parameter, and consequently do not contribute to exponential instability. Moreover, it is known, e.g., [Katok and Hasselblatt 1995, §17.6], that the evolution of any parallel component $z_{\parallel}$ decouples from that of the remaining *normal* one $z_{\perp}$; then, as interested in the possibly exponential growth of the deviation, we are left with only $\mathcal{N}-1$ independent orientations of $\boldsymbol{\nu}$, once fixed the geodesic. Said in other words, we consider only perturbations orthogonal to the geodesic, so that

$$g_{ab}\nu^a\nu^b = 1 \quad ; \quad g_{ab}\nu^a u^b = 0\ ,$$

which leads to

$$\langle\nu^a\nu^b\rangle = \frac{g^{ab}}{\mathcal{N}-1}\ . \tag{9}$$

Within this *reasonable and self consistent approximation*, equation (7) reduces to the averaged one:

$$\frac{d^2 z}{ds^2} + k_{\mathbf{u}}(\mathbf{q})\, z = 0 \tag{10}$$

where

$$k_{\mathbf{u}} = k_{\mathbf{u}}(\mathbf{q}) \stackrel{\text{def}}{=} \frac{1}{\mathcal{N}-1}\mathcal{H}^a{}_a = \frac{1}{\mathcal{N}-1}\mathsf{Tr}\,\boldsymbol{\mathcal{H}} = \frac{1}{\mathcal{N}-1}\sum_{A=1}^{\mathcal{N}-1} K^{(2)}(\mathbf{u},\mathbf{e}_A)\ ; \tag{11}$$

from now on, as above, the dependence on $\mathbf{q} \in \mathsf{M}$ of all geometric quantities will be understood whenever no confusion can arise.
We now sketch the line along which it can be shown, [Cipriani 1993], that this quantity equals the (normalized) *Ricci curvature* in the direction of the flow:

$$k_R(\mathbf{u}) \stackrel{\text{def}}{=} \frac{\mathsf{Ric}(\mathbf{u})}{\mathcal{N}-1} \ , \tag{12}$$

being the Ricci curvature along any direction indicated by the unit vector $\mathbf{e}_A \in \mathsf{T_q M}$, defined as

$$\mathsf{Ric}(\mathbf{e}_{_A}) \stackrel{\text{def}}{=} \ R_{bc} e^b_A e^c_A \ , \tag{13}$$

where $R_{ac} \stackrel{\text{def}}{=} \ R^b{}_{abc} \equiv g^{bd} R_{abcd}$ is the Ricci tensor.
The quantities $K^{(2)}(\mathbf{u}, \mathbf{e}_A)$ in eq.(11) are the sectional curvatures in the $(\mathcal{N}-1)$ independent planes spanned by $\mathbf{u}$ and the $\{\mathbf{e}_A\}$, $(A = 1, \ldots, \mathcal{N}-1)$, $(\mathbf{e}_A, \mathbf{e}_B) = \delta_{AB}$, $(\mathbf{e}_A, \mathbf{u}) = 0$. In general, given a riemannian manifold, $(\mathsf{M}, \mathbf{g})$, the definition of sectional curvature in the plane spanned by two non parallel vectors, $(\mathbf{x}, \mathbf{y}) \in \mathsf{T_q M}$ is defined as

$$K^{(2)}(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \ \frac{R_{abcd} x^a y^b x^c y^d}{\|\mathbf{x}\|^2 \ \|\mathbf{y}\|^2 - \|\mathbf{x} \cdot \mathbf{y}\|^2} \equiv K^{(2)}(\mathbf{y}, \mathbf{x}) \ ; \tag{14}$$

where the norm and the scalar product are those induced by $\mathbf{g}$.
Without loss of generality, [Synge and Schild 1978], we can suppose $\mathbf{x} \perp \mathbf{y}$; so, given an $\mathcal{N}$-dimensional manifold, by simmetry it follows that there are $\mathcal{N}(\mathcal{N}-1)/2$ independent sectional curvatures. Further, given an orthonormal basis in $\mathsf{M}$, $\{\mathbf{e}_{_A}\}$, $(A = 0, \ldots, \mathcal{N}-1)$, we indicate these sectional curvatures as

$$K^{(2)}_{_{AB}} \stackrel{\text{def}}{=} \ K^{(2)}(\mathbf{e}_{_A}, \mathbf{e}_{_B}) \equiv R_{abcd} \ e^a_{_A} e^b_{_B} e^c_{_A} e^d_{_B} \equiv K^{(2)}_{_{BA}}, \quad (B \neq A). \tag{15}$$

We preliminarly show that the *Ricci curvature* along the direction $\mathbf{e}_{_A}$ defined above, eq.(13) is also equal to

$$\mathsf{Tr}\mathcal{R}^{[A]} = \mathcal{R}^{[A]}{}_{bd} \sum_{B=0}^{\mathcal{N}-1}{}' e^b_{_B} e^d_{_B} \tag{16}$$

where we defined the $\mathcal{N}$ tensors $\mathcal{R}^{[A]}{}_{bd} \stackrel{\text{def}}{=} \ R_{abcd} \ e^a_{_A} e^c_{_A}$ and the prime stands to indicate that the sum is over $B \neq A$. Indeed, in order to see that the r.h.s. of eq.(16) actually equals the trace of the corresponding tensor, we observe that, by definition,

$$\mathcal{R}^{[A]}{}_{bd} \sum_{B=0}^{\mathcal{N}-1}{}' e^b_{_B} e^d_{_B} = R_{abcd} \ e^a_{_A} e^c_{_A} \sum_{B=0}^{\mathcal{N}-1}{}' e^b_{_B} e^d_{_B} \ ;$$

and, using the anti-simmetry properties of the Riemann tensor, we can include in the summation also the term $B = A$, which vanishes identically.
Being $\{\mathbf{e}_{_A}\}$ an orthonormal basis, we then have

$$\sum_{B=0}^{\mathcal{N}-1} e^b_{_B} e^d_{_B} = g^{bd} \ , \tag{17}$$

so that:

$$R_{abcd} \ e^a_{_A} e^c_{_A} \sum_{B=0}^{\mathcal{N}-1} e^b_{_B} e^d_{_B} = \mathcal{R}^{[A]}{}_{bd} \ g^{bd} \equiv \mathcal{R}^{[A]b}{}_b = \mathsf{Tr}\boldsymbol{\mathcal{R}}^{[\mathbf{A}]} \ = g^{bd} R_{abcd} \ e^a_{_A} e^c_{_A} = R_{ac} \ e^a_{_A} e^c_{_A} = \mathsf{Ric}(\mathbf{e}_A) \ ; \tag{18}$$

in such a way that the Ricci curvature along any vector of the unitary tangent space coincides with the trace of the corresponding $\boldsymbol{\mathcal{R}}$ tensor. Incidentally, from the leftmost hand side of eq.(18) we see

$$\mathsf{Ric}(\mathbf{e}_A) = \sum_{B=0}^{\mathcal{N}-1}{}' K^{(2)}_{_{AB}} \ ; \tag{19}$$

that is enough to justify also the last equality in eq.(11). Now, given a geodesic passing through $\mathbf{q} \in \mathsf{M}$, let $\mathbf{u} \in \mathsf{T_q M}$ its unitary tangent vector. If we put $\mathbf{e}_{_0} = \mathbf{u}$, we see that what we called *stability tensor* for the given geodesic is actually

$$\mathcal{H}^a{}_b \equiv \mathcal{R}^{[0]a}{}_b \ , \tag{20}$$

and this completes the proof that $\mathsf{Tr}\boldsymbol{\mathcal{H}} \equiv \mathsf{Ric}(\mathbf{u})$, i.e., $k_{\mathbf{u}} = k_R(\mathbf{u})$.
As we have just seen, to define the $(\mathcal{N}-1)$ sectional curvatures when one direction, $\mathbf{u}$, has been fixed as the tangent vector to a given geodesic it suffices to choose the remaining $(\mathcal{N}-1)$ orthogonal unit vectors $\{\mathbf{e}_{_A}\}$, $(A = 1, \ldots, \mathcal{N}-1)$, transversal to the flow. A preferred set is however obtained if we furthermore introduce the *principal sectional curvatures* for the congruence defined by $\mathbf{u}$, i.e., the *local* eigenvalues, $\{\lambda_{_A}\}$, of the stability tensor such that

$$\mathcal{H}^a{}_c\, \mathbf{e}_{_A}{}^c = \lambda_{_A} \mathbf{e}_{_A}{}^a, \qquad (A = 0, \ldots, \mathcal{N}-1) \tag{21}$$

with normalized eigenvectors, $\|\mathbf{e}_{_A}\| = 1$ ; so, by eq.(8), it turns out

$$\lambda_{_0} = 0 \qquad \text{if} \qquad \mathbf{e}_{_0} = \mathbf{u}\ . \tag{22}$$

and also that the Ricci curvature per degree of freedom $k_R(\mathbf{u})$ represents the average of the non trivial principal sectional curvatures $\{\lambda_{_A}\}$[7]. From this last definition, it follows in a perhaps more direct way the well known result, `e.g.`,[Synge and Schild 1978], according to which the sum of all the sectional curvatures in the $(\mathcal{N}-1)$ 2-planes containing a fixed direction do not depends on the choice of the set of independent normal directions, being simply the trace of the tensor $\boldsymbol{\mathcal{R}}^{[A]}$. In principle, the $(\mathcal{N}-1)$ non trivial eigenvalues of $\boldsymbol{\mathcal{H}}$ form a set of indipendent *local* indicators of stability, in what they approximately determine the *local* behaviour of a deviation vector along the corresponding eigendirection; indeed, the analysis carried out for few dimensional systems, [Cipriani and Di Bari 1997c], has shown that the connection between the sectional curvatures and the dynamical behavior do exist, although it emerges completely when considered globally, and that estimates based only on *naïve* local analysis can lead only to partial answers. It has been claimed that there is a trivial counterexample to this, represented by the flows on manifolds of constant curvature. But it isn't at all a counterexample, as, in that case, *Schur's theorem*, [Synge and Schild 1978], assures that there is no distinction between *local* and *global* features of the manifold[8]. So, although it is generally true that the qualitative behaviour of geodesics depends on global rather than local features (and this is the reason of the general failures of *Toda-like* criteria) within the GDA we feel the perception of the intermingled relationships amongst them, which allows to gain some *global* informations from a *local* analysis of the curvature proprties of the manifold.

## Curvature, Instability and Statistical Behaviour.

It has repeatedly been claimed that the chaotic *and* statistical properties of dynamical systems, in the case of mdf , `e.g.`,[Gurzadyan and Savvidy 1986, Kandrup 1990], and for few dimensional systems too, [Szydlowski and Krawiec 1993], depend on the *scalar curvature* of the manifold. Instead, it has been shown with concrete applications, see, for example, [Pettini 1993] or [Cipriani 1993], that this quantity gives no information at all on the behaviour of trajectories for mdf systems, moreover, using the Eisenhart geometrization, it is found to vanish identically, resulting in general a very scarcely reliable indicator. The reasons of this failure in both few and many dimensional DS's turn out easily if we think over the assumptions implicitly or explicitly made to justify its use.
The use of scalar curvature has been accounted for neglecting the orientation of the velocity $\mathbf{u}$, so averaging on *different states* of the system, in addition to different orientations of deviation. In [Cipriani 1993] the consequences of such an approximation are analysed thoroughly; here we show the formal derivation and critically analyse what would be its implications.
If also the tangent vector to the geodesic $\mathbf{u}$ is oriented randomly, as $\|\mathbf{u}\| \equiv 1$ too, we can write, in analogy with what has been done above for $\boldsymbol{\nu}$, except that now we really have $\mathcal{N}$ independent orientations (and then $\mathcal{N}-1$ for the orthogonal deviation vector):

$$\langle u^a u^b \rangle = \frac{g^{ab}}{\mathcal{N}} \tag{23}$$

With this strong assumption, we see that the stability tensor loses its dependence on the *actual state*, forgetting the information about the orientation of the geodesic, and the first term in the r.h.s. of eq.(7) becomes

$$\mathcal{H}_{ac}\nu^a\nu^c \approx R_{abcd}\frac{g^{bd}}{\mathcal{N}}\frac{g^{ac}}{(\mathcal{N}-1)} = \frac{\Re}{\mathcal{N}(\mathcal{N}-1)} \tag{24}$$

[7]As we observed, from eq.(8) it follows that the non trivial geodesic deviations should have a component orthogonal to the flow, as a parallel perturbation cannot contribute to the possible instability. Stated otherwise, the knowledge *a priori* of a vanishing eigenvalue of $\boldsymbol{\mathcal{H}}$ suggests to divide the trace by $(\mathcal{N}-1)$ instead than $\mathcal{N}$; this is also consistent with the fact that, given a direction, there are only $(\mathcal{N}-1)$ independent 2-planes containing it. As we were interested in mdf systems, we wouldn't need here to pursue further this point, however taking into account such a distinction below, when we average also on the orientations of $\mathbf{u}$, counting correctly the number of (ordered) direction pairs as $\mathcal{N}(\mathcal{N}-1)$; also because it has to be remarked the great importance held by this factor in the applications of the GDA to few dimensional DS's.

[8]We will often return in the sequel to some underestimate implications of this theorem, which results of fundamental importance to shed light on the properties of the manifold really tied up to the onset of Chaos.

where we introduced the scalar curvature of the manifold

$$\Re \stackrel{\text{def}}{=} R^a{}_a \equiv g^{ac} R_{ac} \equiv g^{ac} g^{bd} R_{abcd} \tag{25}$$

which, using the previous formulae, turns out also to be

$$\Re \equiv \sum_{A=0}^{\mathcal{N}-1} \mathsf{Ric}(\mathbf{e}_{_A}) = \sum_{A=0}^{\mathcal{N}-1} \sum_{B=0}^{\mathcal{N}-1}{}' K^{(2)}_{_{AB}} = \sum_{A=0}^{\mathcal{N}-1} \mathsf{Tr}\mathcal{R}^{[A]} \tag{26}$$

and where we averaged over the right number of directions *pairs*. Using these approximations, equation (10) is replaced by

$$\frac{d^2 z}{ds^2} + k_S \; z = 0 \tag{27}$$

where the average scalar curvature has been defined, $k_S \equiv k_S(\mathbf{q}) \stackrel{\text{def}}{=} \dfrac{\Re}{\mathcal{N}(\mathcal{N}-1)}$.
We see from the derivation itself that the use of scalar related quantities is unjustified, because the averaging over directions of geodesics means to ignore the actual state of the system; i.e., the evolution of the perturbation to a *given geodesic, along the geodesic itself* results to depend on the paths associated to all the geodesics passing on the same point $\mathbf{q}$, with arbitrary velocity! This is the same as saying that this claimed *average over states* turns out to be instead an average over $\mathcal{O}(\mathcal{N}^2)$ evolutionary equations amongst which only $\mathcal{O}(\mathcal{N})$ carry meaningful information, which turns out obviously suppressed by the $\mathcal{O}(\mathcal{N}^2 - \mathcal{N})$ irrelevant ones.
We will show below that performing that average amounts to assume that the manifold is isotropic; this is definitely not true for whatever realistic model of physical DS, whereas it is exactly what happens for abstract geodesic flows of mathematical ergodic theory! The only physically meaningful situation in which the scalar curvature carries the correct information is that of two dimensional manifolds, i.e., the case in which there is only one sectional curvature, equal to the Ricci one, in turn equal to half times the scalar curvature, [Synge and Schild 1978]. So, any averaging process must be carried out carefully, if reliable results are sought. This is true in general, but in particular, for few degrees of freedom systems, there are two opposite situations and a list of warnings is mandatory:

- As remarked above, for two dimensional configuration spaces, all the curvatures are directly related each other and the results obviously do not depend on the choice made.

- This do not means that for few (but $\mathcal{N} > 2$) degrees of freedom, the scalar curvature should give reliable results. Even more, in these cases, the global constraints in the geometrical features of the manifold causes any averaging to destroy the information contained in the sectional curvatures, causing even the Ricci one to lead to wrong results, as the averaging can hide all the mechanisms responsible of the onset of Chaos, [Di Bari 1996, Cipriani and Di Bari 1997a].

- To highlight the peculiarities of two-dimensional manifolds, we remind that for DS's with two degrees of freedom, if interested only in the *phenomenological* determination of dynamic (in)stability, no matter what geometrization is adopted, the results are in agreement. But if the goal is the discovery of the origin of Chaos, it should be taken into account that a suitable enlargement of the manifold gives deeper insights, [Cipriani and Di Bari 1997c]. This is not necessary if the manifold is already three-dimensional or more, [Di Bari and Cipriani 1997c].

- To point out the general (for any $\mathcal{N}$) unreliability of scalar related quantities, and even of Ricci curvature in the case of small $\mathcal{N}$, we recall some results obtained within the Eisenhart geometrization framework, [Pettini 1993]: for any conservative DS, $\Re \equiv 0$, $\mathsf{Ric}_E(\mathbf{u}) \equiv 2$ even for a paradigmatic chaotic hamiltonian (the Hénon-Heiles system). So, eq.(27) turns out to be always meaningless, and even eq.(10) is completely unreliable for two degrees of freedom Hamiltonians.

- Also within the Finsler GDA, it is shown that for a three dimensional manifold (corresponding to a two degrees of freedom DS, [Cipriani and Di Bari 1997c]), even the use of $\mathsf{Ric}(\mathbf{u})$ leads to incorrect answers.

- The predictions on the N-body system based on the scalar curvature are in complete disagreement with those obtained using Ricci curvature, which instead agree with those of standard tools of investigation of chaotic properties.

- The last points relate again to an hasty extension of results borrowed from Ergodic theory of abstract DS's and to an inadequate consideration of *Schur's theorem*. Indeed it has become more and more evident that the properties of manifolds associated with realistic physical models differ significantly from those of manifolds studied in Ergodic theory and that the instability, when present, has very different sources in the two classes of geodesic flows.

- In the latter class, actually, the mechanism of instability resides in the negativity of the sectional curvatures; if *all* the sectional curvatures are *always* negative, then the flow is exponentially unstable, the system is mixing and all the statistical properties are well justified, [Katok and Hasselblatt 1995]. Again, in that case also the Ricci and scalar curvatures are *always* negative, and all approximated/averaged equations predict instability as the exact ones do.

- Conversely, the sources of instability for physically meaningful geodesic flows are instead related to the non constant and anisotropic features of the manifold. In such a case whatever averaging process risk to destroy both non uniform properties, which are moreover each other tightly related via the Schur's theorem. For mdf systems simple statistical considerations, actually the use of the *Central Limit theorem*, allow to recover the fluctuations (both in time and directions) and to obtain very accurate answers on the global behaviour, [Cipriani 1993, Casetti and Pettini 1995, Casetti et al. 1995]. But, when such statistical arguments are not justified by low $\mathcal{N}$ values, the use of averaged quantities change completely the answers.

- Incidentally, we remind that the GDA requires some obvious conditions to be applied. We refer to [Cipriani 1993, Di Bari 1996, Di Bari et al. 1997, Di Bari and Cipriani 1997b] for an exhaustive discussion, and recall here the apparent singularity of the Jacobi line element, $ds_J$, eq.(1), at the *turning points*, where $\mathcal{U}(\mathbf{q}) = E$. For a *standard* DS, whose kinetic energy is a positive definite quadratic form and then these turning points are only on the boundary of the region allowed to motion, it is well known that this do not result in any *real* singular behavior, and, moreover, for mdf systems, the chance that the kinetic energy vanishes becomes smaller and smaller with the increase of $\mathcal{N}$. Nevertheless, the Jacobi geometrodynamics has been applied, [Szydlowski and Krawiec 1993], to a three dimensional General Relativistic DS, for which the *kinetic energy* is not positive definite. In that case the singularity of the metric is *inside* the region allowed to motion and the Jacobi form of the GDA cannot be applied. This is one of the two main points which invalidate the results there obtained, in addition to the use of the scalar curvature, whose general lack of reliability (extolled for two dimensional DS's) has already been pointed out. It is just in order to overcome this kind of limitations of the Jacobi GDA, that we proposed the Finsler extension whose application to Celestial Mechanics and General Relativistic DS's has been performed succesfully, [Di Bari and Cipriani 1997a, Di Bari and Cipriani 1997c].

Summarizing, we can say that, while the use of Ricci related quantities is trustworthy for mdf systems, and for small $\mathcal{N}$ no averaging procedure is guaranteed to give reliable answers, the scalar curvature has in general, as it is, no relationship with the issue of instability, except in the very exceptional situations in which it carries the same informations of the Ricci or sectional curvatures.

## Sources of geodesic instability.

From now on we concentrate on the GDA to Chaos in mdf Hamiltonian systems; therefore we will assume that $\mathcal{N} \gg 1$, so we can neglect the full set of equations (4) and limit ourselves to the averaged eq.(10).
We are left with a formally simple ordinary differential equation for the norm of the perturbation, $z(s)$:

$$z'' + \chi(s)\, z = 0 \,, \tag{28}$$

where $(\cdot)' = d(\cdot)/ds$. According to the level of approximation, the quantity $\chi(s)$ entering in the evolution of $z(s)$ would be, in turn:

$$\chi(s) \equiv \tilde{\chi}\,[\mathbf{q}(s), \mathbf{u}(s)] = \begin{cases} k_R(s) = \tilde{k_R}[\mathbf{q}(s), \mathbf{u}(s)] & = \dfrac{\mathsf{Ric}(\mathbf{u})}{\mathcal{N}-1} \\[2ex] k_S(s) = \tilde{k_S}[\mathbf{q}(s)] & = \dfrac{\Re}{\mathcal{N}(\mathcal{N}-1)} \end{cases} \tag{29}$$

From the preceding discussion, it is clear that we will not consider the second case in eq.(29), except when it is practically equivalent to the first. Rather, we remind that another set of $(\mathcal{N}-1)$ equations which, although being

approximated, give more detailed informations on the stability of the flow, could be obtained letting the *frequency* $\chi(s)$ to assume the value of every non trivial eigenvalue of $\mathcal{H}^a{}_c$, $\chi(s) = \lambda_A(s)$, with $A = (1, \ldots, \mathcal{N}-1)$. For systems with few degrees of freedom the equations so obtained, yet approximated, have proven to reproduce faithfully the qualitative behaviour of solutions of the exact geodesic deviation equations, even when the Ricci curvature fails (nothing to say about the scalar curvature), [Cipriani and Di Bari 1997c]. For high dimensional DS's the problem stands in the huge computational work required to diagonalize $\boldsymbol{\mathcal{H}}$, rather than in the integration of $(\mathcal{N}-1)$ equations instead of one. In some cases, however, the stability tensor turns out to be easily diagonalizable, [Di Bari and Cipriani 1997c], and the resulting equations help to get some hints on the reliability of the results obtained within the *Ricci-averaged* approach. Equation (28) is our starting point for the discussion of the links between geometry and instability. As it stands, it points out the importance of the curvature properties of the manifold on the evolution of a generic perturbation to a geodesic flow. Later on, we will discuss the steps needed to rephrase the results in terms of the newtonian time $t$; for the time being we limits to general considerations on the stability of the geodesic flow, described as the Jacobi $s$-parameter evolves.

Most of the results of abstract Ergodic theory refer to geodesic on manifolds of *constant negative sectional curvature*, [Anosov 1967, §6.1]. In this case, i.e., when $\lambda_A(s) \equiv -\alpha^2 = \text{const.}$ , $\forall\, A = 1, \ldots, (\mathcal{N}-1)$, whatever choice is made in eq.(29), it is apparent that it is also

$$\chi(s) \;\equiv\; -\alpha^2 \;=\; \text{const.} \tag{30}$$

Well then, eq.(28) implies that the magnitude of the deviation will increase exponentially fast in $s$:

$$z(s) = z_o \cosh(\alpha s) + \frac{z_o'}{\alpha}\sinh(\alpha s) \underset{|s|\to\infty}{\longrightarrow} C\ \exp\alpha|s|\ . \tag{31}$$

In this case it has been *rigorously proved*, `e.g.`,[Katok and Hasselblatt 1995], that the dynamics possess the strongest statistical properties, as positive Kolmogorov entropy, mixing, exponential decay of correlations, and so on; and it is in fact the *most chaotic* one. Therefore, if a DS were characterized by *constant negative sectional curvatures*, then it would be strongly chaotic and approaching the equilibrium state within a *relaxation time* of order $S_r \approx \alpha^{-1}$, or, in terms of the newtonian time[9],

$$T_r \approx (\langle\mathcal{W}\rangle_t\ \alpha)^{-1}\ . \tag{32}$$

As far as we know, in the case of *non constant* but *everywhere negative* sectional curvatures,

$$\lambda_A(s) \le -\beta^2 < 0, \quad \forall A = 1, \ldots, (\mathcal{N}-1); \qquad \lambda_0 = 0\ , \tag{33}$$

while the flow is definitely unstable, [Katok and Hasselblatt 1995], for what concerns the statistical properties, it is plausible, and there are convincing argumentations, according to which also the approach to equilibrium take place on a $s$-time scale of order $\beta^{-1}$, although we are not aware of any rigourous and explicit theorem.
Bearing in mind that the exact equation for the evolution of geodesic deviation is eq.(7), of which eq.(28) is an approximation, we should discuss at this point the relevance of Schur's theorem (see, `e.g.`, [Synge and Schild 1978, §4.1]), which refers to *isotropic* Riemannian manifolds[10], whose definition is the following:

> "An $\mathcal{N}$-dimensional ($\mathcal{N} > 2$) riemannian manifold, $\mathsf{M}_{\mathcal{N}}$, is said isotropic at a point $\mathbf{q} \in \mathsf{M}_{\mathcal{N}}$ if the sectional curvatures $K^{(2)}(\mathbf{x}, \mathbf{y})$ evaluated at $\mathbf{q}$, do not depend on the pair $(\mathbf{x}, \mathbf{y})$, i.e., on the *2-plane*."

In this case the Riemann tensor has the form

$$R_{abcd} \;=\; \mathcal{K}\ (g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{34}$$

where $\mathcal{K}$ is a scalar quantity, a priori depending on the position, $\mathcal{K} = \mathcal{K}(\mathbf{q})$. Well, then the Schur's theorem assures that:

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[9] A more careful evaluation of the *physical* relaxation time would give the result

$$\frac{\langle\mathcal{W}\rangle_t}{\langle\mathcal{W}^2\rangle_t} \lesssim \alpha\cdot T_r \lesssim \langle\mathcal{W}\rangle_t^{-1}\ .$$

Although the two limits do not differ significantly for *standard* mdf systems, as the FPU chain, in the case of very inhomogeneus gravitational N-body system out of *global virial equilibrium*, the difference can be quite remarkable. This has no consequences in this context because both them have neither constant or negative sectional curvatures. The implications of these more accurate limits will be discussed elsewhere.

[10] The extension to more general differential manifolds, as the Finsler ones, is discussed in [Rund 1959].

"If an $\mathcal{N}$-dimensional ($\mathcal{N} > 2$) riemannian manifold, $M_{\mathcal{N}}$, is isotropic in a region, then the riemannian curvature is constant throughout that region"

That is, the scalar $\mathcal{K}$ is a constant quantity, do not depends on $\mathbf{q}$. This incidentally implies also that the covariant derivative of Riemann tensor vanishes.
What this theorem entails on the issues we are facing on can be deduced looking rather at its *reverse* implications. Indeed, according to the theorem, if a manifold is isotropic, then *all* the sectional curvatures are also constant; nevertheless, if the manifold is anisotropic, nothing can be said about the constancy or variability of the sectional curvatures, as it could be either[11]

$$K^{(2)}(\mathbf{x}_1, \mathbf{y}_1) = \alpha_1 = \text{const.} \neq K^{(2)}(\mathbf{x}_2, \mathbf{y}_2) = \alpha_2 = \text{const.} \;,$$

or $K^{(2)}(\mathbf{x}, \mathbf{y}) = A(s)$, a variable quantity; and in this case the theorem do not give more informations. Then, this occurrence can be interpreted saying either that $\lambda_A \neq \lambda_B$ in consistent with both variable or constant $\{\lambda_A\}$, or, conversely, that $\lambda_A = \text{const.}$, $\forall\ A$ is consistent with both isotropy or anisotropy.
Other interesting *back implications* of the theorem explain what can be learned looking at the behaviour of Ricci curvature.

- If *at least one* of sectional curvatures varies, $\lambda_A = \lambda_A(s)$, then the manifold cannot be isotropic.
- For an isotropic manifold, i.e. for $\lambda_A \equiv \alpha$, $\forall A = 1, \ldots, (\mathcal{N} - 1)$, then also the Ricci and scalar curvatures are also constant, and $k_R = k_S = \alpha$. This is the sole situation in which the averaged quantities equal exactly the the sectional curvatures.
- If the manifold is anisotropic, then the Ricci curvature along different geodesics assumes distinct values, whereas the scalar curvature do not distinguish between various initial conditions.
- Even in an anisotropic manifold, if the sectionals are constant, then both $k_R$ and $k_S$ do not vary. In this case they represents their averaged values.
- When at least one of the sectionals varies, the manifold must be anisotropic, and in this case the averaging procedure can hide the fluctuations: with fluctuating $\{\lambda_A(s)\}$, it is possible to have either $k_R(s)$ also varying with $s$, or even constant. These remarks apply with even more relevance for scalar related quantities, [Cerruti-Sola and Pettini 1995].
- In particular, if we find that $k_R = \text{const.}$, we cannot say nothing about the behaviour of sectional curvatures, because this is consistent with all the possible features, namely, we can have either an isotropic or an anisotropic manifold with constant sectionals, or even an anisotropic one with fluctuating $\lambda_A$'s!
- Vice versa, if the outcome is that $k_R(s)$ is a fluctuating quantity, we can definitely assert that also the sectional curvatures vary along the geodesic, and, by the Schur's theorem, that the manifold is actually anisotropic.

*Luckily enough*, almost all physically meaningful geodesic flows fall into this last category, and either farther from complete integrability we are or higher is $\mathcal{N}$, more true it is. According to this picture, the manifold is anisotropic, with sectional curvatures rapidly fluctuating, almost independently from each other, in such a way that, when $\mathcal{N} \gg 1$, we can assume (and even verify!) that $\mathsf{Ric}(\mathbf{u})$ is the sum of $\mathcal{N} - 1$ uncorrelated quantities, of which $k_R(\mathbf{u})$ represents the average. For $\mathsf{mdf}$ systems, and when the motion is definitely far from quasi-periodicity, it can be shown, [Cipriani 1993, Casetti and Pettini 1995, Casetti et al. 1995], that it is consistent also to assume the random character of the quantities entering $\mathsf{Ric}(\mathbf{u})$, and using the central limit theorem, to relate the fluctuations of $k_R(\mathbf{u})$ to those of the $\lambda_A$'s, obtaining an analytical estimate of the asymptotic rate of growth of the norm of the deviation vector, i.e., of the $\mathsf{LCN}$.
Once realized that the case in which *all* the sectional curvatures are *constant* is an exceptional one, and consequently that the rigorous results on abstract geodesic flows carry very little physical interest, we turn to the study of the evolution of a perturbation to a realistic geodesic flow.

[11] In the sequel we will rephrase these argumentations in terms of the principal sectional curvatures along the flow, (the $\lambda_A$'s), though these depend on a parallely propagated vector. This is legitimate as long as $\boldsymbol{\mathcal{H}}$ satisfies very generic hypotheses. A very interesting account on these general arguments can be found in [Berndt and Vanhecke 1992].

## Mechanisms responsible of the onset of Chaos.

Bearing in mind the warnings made above, we now look at eqs.(28) and (29), in order to see under what conditions instability can arise. It is obvious that when $\chi(s) = -\alpha^2 = \text{const.}$, these equations predict an exponential growth of the magnitude of the geodesic deviation. As stressed before, a constant value of $\mathsf{Ric}(\mathbf{u})$ can occur for very different behaviour of sectional curvatures, but a negative constant value implies that the average value of them is permanently negative, and this do not happen for any geodesic flow of physical significance, neither it found that $k_R$ is mostly negative. To see this, let us write the expressions for Ricci and scalar curvature in the Jacobi metric for a conservative DS with $\mathcal{N}$ degrees of freedom, [Cipriani 1993]:

$$k_R(\mathbf{u}) = \frac{1}{2\mathcal{N}\mathcal{W}^2}\left\{\Delta\mathcal{U} + \frac{(\mathrm{grad}\,\mathcal{U})^2}{\mathcal{W}} + (\mathcal{N}-2)\left[\frac{1}{2}\left(\frac{d\mathcal{U}}{ds}\right)^2 + \mathcal{W}\frac{d^2\mathcal{U}}{ds^2}\right]\right\} , \tag{35}$$

and

$$k_S(\mathbf{u}) = \frac{1}{\mathcal{N}\mathcal{W}^2}\left[\Delta\mathcal{U} - \left(\frac{\mathcal{N}-6}{4}\right)\frac{(\mathrm{grad}\,\mathcal{U})^2}{\mathcal{W}}\right] , \tag{36}$$

where we indicated the usual $\mathcal{N}$*-euclidean* Laplacian and gradient operators with $\Delta$ and grad, respectively. A first look at the expressions shows that, on passing from the average over directions of the deviation vector to the loose approximation given by the scalar curvature, some important information get lost. Indeed in both $k_R$ and $k_S$ appear the $\mathcal{N}$-dimensional laplacian of the interaction potential, with a weight reduced by a factor 2 in the former, and the $\mathcal{N}$-dimensional squared gradient, i.e., the sum of the squared forces acting on each *particle* of the system. One main difference is on the sign and on the weight of this term, which is amplified by a factor $\mathcal{O}(\mathcal{N})$ in the scalar related frequency, $k_S$ to which it contributes negatively, thus increasing the chance of trivial instability of the solution. At least equally relevant to the *evolution of the instability* is the lacking in $k_S$ of the last two terms in $k_R$, as they describe the *explicit* $s$-time dependence of the curvature of the manifold, in particular for systems out of *global virial equilibrium*, i.e., subject to *collective oscillations* which involve macroscopic fluctuations of total potential and kinetic energies, as it occur for a collisionless N-body system during its very early stages. The consequences of the differences just listed can be grasped intuitively if rephrased in terms of the newtonian $t$-time. As a matter of fact, we see that the last two terms in the r.h.s. of eq.(35) are nothing else than the *logarithmic time derivatives* of the kinetic (or potential) energy:

$$\frac{d\mathcal{U}}{ds} = \frac{1}{\sqrt{2}}\frac{\dot{\mathcal{U}}}{E-\mathcal{U}} = -\frac{1}{\sqrt{2}}\frac{\dot{\mathcal{W}}}{\mathcal{W}} \quad ; \qquad \frac{d^2\mathcal{U}}{ds^2} = -\frac{1}{2\mathcal{W}}\frac{d}{dt}\left(\frac{\dot{\mathcal{W}}}{\mathcal{W}}\right) = -\frac{1}{2\mathcal{W}}\left[\frac{\ddot{\mathcal{W}}}{\mathcal{W}} - \left(\frac{\dot{\mathcal{W}}}{\mathcal{W}}\right)^2\right] \; ; \tag{37}$$

and, for a mdf system, these relative fluctuations are damped by a statistical factor that obviously increases with $\mathcal{N}$. Moreover, they exploit an explicit dependence of $k_R(s)$ on rapidity with which the *global virial equilibrium* is attained. The magnitude of those relative fluctuations indeed, though generally smaller and smaller as the number of degrees of freedom grows, depends in general on the *virial ratio*, and could be not negligible, in particular for a self-gravitating system, where collective effects are likely to occur, due to the long range nature of the interaction. Thus, whereas the Ricci curvature keeps memory of these evolutionary processes, the scalar one, just because forget *ab initio* the real dynamics of the flow, erasing the $\mathbf{u}$ dependence, cannot take care of the consequences of the approach to this dynamical global equilibrium. If we add to these remarks the presence of $(\mathcal{N}-1)$ *Ricci curvatures* along directions extraneous to the dynamics, the *wrong* sign and weight it attributes to the square average force[12], with respect to Ricci curvature in both Jacobi and Finsler cases, [Cipriani 1993, Di Bari 1996, Di Bari et al. 1997], together with the evidence of the effectiveness of the use of the latter in gaining deep insights on the sources of instability and in the analytical computation of instability exponents, [Casetti et al. 1995, Cipriani 1993, Cipriani and Di Bari 1997c], it turns out once more why the use of scalar related quantities is ever more unreliable in the case of mdf systems.
A first look to the expression of the average curvature in the $(\mathcal{N}-1)$ 2-planes containing $\mathbf{u}$ leads to the following considerations:

- The $\mathcal{N}$-dimensional laplacian is almost everywhere positive for every confining interaction. It is always positive near the minimum of any potential and can be locally negative only in limited regions.
- The squared $\mathcal{N}$-gradient contributes with a positive sign to $k_R$.

[12] Which has been responsible of a largely unjustified debate in the past, [Gurzadyan and Savvidy 1986, Kandrup 1990].

- For $\mathcal{N} > 2$ also the square of the of the $s$-time derivative of the potential energy contributes with a plus sign.
- Then, the term with the second derivative of $\mathcal{U}$ alone can give a negative contribution to the Ricci curvature.

What it has been found for the FPU chain, either numerically integrating the trajectories, [Pettini 1993] and [Cipriani 1993], or canonically averaging over phase space, [Casetti and Pettini 1995] is that the Ricci curvature (and then $k_R$) is mostly positive, that its average value is *always* positive, that there is no correlation at all with the degree of stochasticity in the dynamics (as measured by, `e.g.`, the maximal LCN),[Cipriani et al. 1996], that the chances to find a negative value for it decrease quickly increasing $\mathcal{N}$, vanishing in the thermodynamic limit. Moreover, in [Cipriani 1993], it has been found the numerical evidence of the actual occurrence of the foreseen *virial transition* process, [Cipriani and Pucacco 1994a, Cipriani and Pucacco 1994b]. In the figures 1 and 2, this transition is neatly detected, as well as its dependence on the number of degrees of freedom and on the regime of chaoticity. It is found that, while during the first phase of *virialization*, the frequency of events such that $k_R < 0$ is not negligible, after then, and already for relatively small values of $\mathcal{N}$ ($\sim 10^2$), the probability $\wp$ of a negative value for $k_R$, is actually vanishing,

$$\wp(k_R < 0) \sim C_{\mathcal{N}} \exp\left[-(t/T_v)^b\right] \xrightarrow[t > T_v]{} 0 \; ; \tag{38}$$

as highlighted, from the *pure* $t^{-1}$ behaviour of the *measured cumulative frequency*, $F_-(t)$[13]:

$$F_-(t) \stackrel{\text{def}}{=} \frac{N_-(t)}{N(t)} = \frac{N(k_R < 0)}{N_{\text{steps}}} \tag{39}$$

shown in figure 1. We observe, moreover the strong decrease of this frequency with $\mathcal{N}$, and the very weak correlation with the degree of stochasticity, as shown by figure 3. The following discussion explain the qualitative behaviours of the constant $C_{\mathcal{N}}$ and the virialization time $T_v = T_v(\mathcal{N}, \beta\epsilon)$. When it was firstly detected, it was surprising enough to see the lack of correlation between the degree of stochasticity, as measured by the maximal LCN, or the parameter $\beta\epsilon$, and the frequency of occurrence of negative values. Even the $\mathcal{N}$ dependence was contrary to what expected on the basis of the belief that the statistical description is more reliable as the number of particles increases. On the light of the analytical estimates, since then these results turn out however completely understandable. A thorough analysis of these results is presented in [Cipriani 1993] and completed in [Cipriani and Di Bari 1997b], but we can say here that the efficiency of the virialization process is clearly decreased by the occurrence of a quasi periodic behaviour, i.e., by a nearly integrable dynamics. Moreover, the *phase mixing*, which is in turn responsible of the trivial loss of correlations between particles, becomes faster along with the increase of the largest normal mode frequency excited, i.e., with $\mathcal{N}$ too.

As the unpredictability is believed to growth with the numbers of interacting particles, the inverse correlations between $\mathcal{N}$ and $\wp(k_R < 0)$, once more force us to discard the hypotesis of an instability driven by negative values of Ricci curvature. Besides, the results obtained within the Jacobi picture are in complete agreement with those coming out from the Eisenhart geometrization, [Casetti and Pettini 1995, Cipriani 1993]; and in this last framework, the Ricci curvature for the FPU chain is always positive (more, it is always $k_{R_{\rm E}} \geq 2$), nevertheless, is there possible to recover in a very elegant and effective way all the (in)stability properties of the dynamics, obtaining an analytical algorithm for computation of the largest LCN.

If the sign of $k_R$ is mostly positive, and seeing that numerical integrations of eq.(28) allow to recover all the qualitative and quantitative stability properties of mdf systems, we are led to ask where stems from the exponential growth of the geodesic deviation. The answer is actually written in any textbook, `e.g.`,[Landau and Lifsits 1980], and resides in the mechanism of the swing: the instability is driven by *parametric resonance* induced by the fluctuations of the positive value of the *frequency* $\chi(s)$. We refer to [Cipriani 1993, Pettini 1993, Casetti and Pettini 1995, Di Bari 1996, Di Bari et al. 1997] for details, and here simply recall that in the non-autonomous equation

$$\ddot{x} + \Omega^2(t) x = 0 \; , \tag{40}$$

with

$$\Omega^2(t) = \Omega_o^2[1 + f(t)] \quad ; \qquad |f(t)| < 1 \quad ; \qquad \langle f(t) \rangle_t = 0$$

instability (i.e., exponential growth) of the solution can arise, fixed the average frequency $\Omega_o$, if suitable conditions are fulfilled by the modulation factor $f(t)$. In the case of eq.(28), the modulating factor is not an explicit function of $s$-time, rather depends on $s$ through the actual *state* $[\mathbf{q}(s), \mathbf{u}(s)]$, of the system. So, in order to foresee the behaviour

[13] Defined as the ratio between the total number of occurrence of negative values of Ricci curvature and the total number of steps performed. That is, $N_{\text{steps}} = t/\Delta t$, $\Delta t$ being the integration time step.

of the deviation vector it is important to know not only the average value $\chi_o \stackrel{\text{def}}{=} \langle\chi(s)\rangle_s$, but also the amplitude, $\sigma_\chi^2 = \langle(\chi(s)-\chi_o)^2\rangle_s$ and the variability time-scale $\tau_s$ of its fluctuations.
Once again, we see why the cancellation of the *fluctuating* terms consequent to the averaging over $\mathbf{u}$, is at the origin of the unreliable results obtained with scalar curvature.

In order to compare the outcomes obtained within the GDA with those resulting from standard dynamical system approach, we need to rewrite the equations in terms of the newtonian $t$-time, exploiting the non affine relation given by eq.(1). If we do this, equation (28) becomes:

$$\frac{d^2z}{dt^2} - \frac{\dot{\mathcal{W}}}{\mathcal{W}}\frac{dz}{dt} + \hat{k}_R z = 0 \ , \tag{41}$$

where $\hat{k}_R \stackrel{\text{def}}{=} 2\,\mathcal{W}^2\,k_R$ is a sort of *rescaled* Ricci curvature. We are left wiht an ordinary differential equation for a *damped* harmonic oscillator, whose peculiarities lie both in the fast variability of the *frequency* $\hat{k}_R(t)$, and in the undefined sign of the *damping* term, which actually has zero average and can act alternatively also as an *anti-damping*. The presence of this term is nevertheless crucial in order to understand the sources of instability in the *physical time gauge*. Indeed, an equation completely equivalent to (41) is obtained with the change of variable

$$Y(t) \stackrel{\text{def}}{=} \mathcal{W}^{-1/2}(t)\ z(t) \ , \tag{42}$$

where it is understood that all the quantities depends on $t$ through the one-to-one correspondence $s = s(t)$, fixed by $ds = \sqrt{2}\,\mathcal{W}\,dt,\ \mathcal{W} > 0$. For mdf systems indeed the probability that $\mathcal{W} = 0$ is negligible, and, for a confined DS, the virial theorem guarantees also that $\mathcal{W}(t)$ is a quasi periodic function, bounded from above[14]; in such a way that the qualitative long term time behaviours of $Y(t)$ and $z(t)$ are the same. This well known substitution allows to get rid of the damping term, leading to a standard *Hill's equation*:

$$\ddot{Y}(t) \ + \ Q(t)\ Y(t) \ = \ 0 \ , \tag{43}$$

where the frequency $Q(t)$ follows from the rescaled average curvature $\hat{k}_R(t)$:

$$Q(t) \equiv \tilde{Q}\{\mathbf{q}(t)\} \stackrel{\text{def}}{=} \hat{k}_R(t) - \frac{3}{4}\left(\frac{\dot{\mathcal{W}}}{\mathcal{W}}\right)^2 + \frac{1}{2}\frac{\ddot{\mathcal{W}}}{\mathcal{W}}. \tag{44}$$

As heralded above, the terms appearing in the expression of $Q(t)$ with respect to $\hat{k}_R(t)$ are very important for the understanding of the behaviour of the perturbations in the *physical* time. To see why, let us write down the explicit $t$-expressions:

$$\hat{k}_R(t) = \frac{1}{\mathcal{N}}\left\{\Delta\mathcal{U} + \frac{(\text{grad}\,\mathcal{U})^2}{\mathcal{W}} + (\mathcal{N}-2)\left[\frac{3}{4}\left(\frac{\dot{\mathcal{W}}}{\mathcal{W}}\right)^2 - \frac{1}{2}\frac{\ddot{\mathcal{W}}}{\mathcal{W}}\right]\right\} \ , \tag{45}$$

$$Q(t) = \frac{1}{\mathcal{N}}\left\{\Delta\mathcal{U} + \frac{(\text{grad}\,\mathcal{U})^2}{\mathcal{W}} + \left[\frac{\ddot{\mathcal{W}}}{\mathcal{W}} - \frac{3}{2}\left(\frac{\dot{\mathcal{W}}}{\mathcal{W}}\right)^2\right]\right\} \ . \tag{46}$$

We see that the terms arising from the $s$ derivatives of the potential energy, and here rewritten in terms of $t$ derivatives, appear in $Q(t)$ with a weight that, in the large $\mathcal{N}$ limit, is actually reduced by a factor $\mathcal{N}^{-1}$. The relevance of this outcome becomes self-evident as we consider the the sources of instability in eq.(43), or (41). Indeed, as long as we consider *customary* mdf systems, as emphasized above, it is *never* found that $Q(t) < 0$, and the frequency with which $\hat{k}_R(t) < 0$, is always very small, becoming absolutely negligible with increasing $\mathcal{N}$, in which case asimptotically *vanishes*, as the virial equilibrium is attained; see figures 1 and 7 and also [Cipriani 1993, §5.2.1]. In addition, *this frequency do not show any correlation with the degree of stochasticity*, as it is evident from figure 3, obtained from [Cipriani 1993], see also [Cipriani et al. 1996].
The occurrence of Chaos, i.e., the exponential growth of all the solutions (except, possibly, for a set of initial conditions of zero measure) of eq.(43) is consistent with a positive sign-definite $Q(t)$ if the mechanism responsible of the onset of instability is the *parametric resonance* mentioned above. It is crucial for this phenomenon to occur that the amplitude

[14]This is not strictly true asimptotically for the *mathematical* gravitational N-body system, but it is true for almost all initial conditions at any finite time. Moreover, and this is an important point, $\mathcal{W}$ is *always bounded* from above, when the *physical* interaction is addressed. We stress that the strategy we introduce below to cope with the mathematical singularities of the gravitational potential *do not change* its very nature of *non-compact* DS.

of fluctuations fulfil some conditions, whose details depend either on their quasi periodicity or on their stochastic behaviour. This explain the relevance of the reduced weights in front to the fluctuating terms in $Q(t)$ with respect to $\hat{k}_R(t)$ in order to recover the known results in terms of the newtonian time.
The argument just discussed enlighten also why a mdf system should go across a *transition* in its qualitative behaviour along the approach to the *global virial equilibrium*, during which the amplitude of the fluctuations in its macroscopic parameters, as the kinetic and potential energies is damped just in consequence of the trivial *phase mixing* previously recalled[15]. When the virialization phase is accomplished, the instability of the dynamics is governed mostly by the fluctuations in the *dominant* term, which, accordingly to a kind of *principle of equivalence of any geometrization* (in any case confirmed by analytical calculations and numerical computations) is the $\mathcal{N}$-dimensional euclidean laplacian, see table 1.

Being the translation of the dynamics in (Jacobi) geometrical language almost straightforward for any additive interaction potential, $\mathcal{U}$, we could achieve order of magnitude estimates of all these quantities in the general case; nevertheless, to avoid a (still) heavier discussion, we will consider separately the FPU unidimensional chain and the gravitational N-body problem[16], that can be considered most representative among the short and long range, respectively, interaction potentials.

## *Standard* potentials: the FPU chain.

The first mdf system investigated by computer simulations has been the celebrated unidimensional chain of N weakly anharmonic oscillators. According to the power-law of the anharmonic terms, cubic or quartic, these models were indicated as FPU-$\alpha$ and FPU-$\beta$ models, respectively, [Fermi et al. 1965]. We will focus our attention here on the second kind, essentially because it has been thoroughly investigated, both from dynamical as well Statistical Mechanical points of view, see, e.g., [Pettini and Cerruti-Sola 1991, Benettin et al. 1984] and references therein.
In this case $d = 1$, so $\mathcal{N} = N$ and the Hamiltonian is

$$H(\mathbf{x},\mathbf{p}) = \sum_{i=1}^{N}\left[\frac{1}{2}p_i^2 + \frac{1}{2}(x_{i+1}-x_i)^2 + \frac{\beta}{4}(x_{i+1}-x_i)^4\right] \tag{47}$$

We recall that, in the quasi-harmonic limit, namely for $\beta\epsilon \ll 1$, where $\epsilon \stackrel{\mathrm{def}}{=} E/N$ is the specific energy ($E \equiv H(\mathbf{x},\mathbf{p})$), the introduction of the *normal modes* allows to exploit the properties of this quasi-integrable system. Such a *good* set of coordinates, $(\mathbf{X},\mathbf{P})$, is defined as, [Toda et al. 1992]:

$$x_i = \left(\frac{2}{N}\right)^{1/2}\sum_{k=1}^{N-1} X_k \sin\left(\frac{ik\pi}{N}\right) \qquad , \qquad P_k = \dot{X}_k \tag{48}$$

and is well suitable for a *perturbative* treatment. The Hamiltonian, eq.(47), in the new coordinates reads

$$H(\mathbf{X},\mathbf{P}) = \sum_{k=1}^{N}\left[\frac{1}{2}({P_k}^2 + {\omega_k}^2{X_k}^2) \;\; + \;\beta \sum_{\{j_1,j_2,j_3\}=1}^{N} C(k,j_1,j_2,j_3)X_kX_{j_1}X_{j_2}X_{j_3}\right] \tag{49}$$

with $\omega_k = 2\sin\left(\frac{k\pi}{N}\right)$ and the $C(k,j_1,j_2,j_3)$ are coefficients depending on the choice of the boundary conditions (fixed or periodic). Using these coordinates the harmonic limit, $\beta\epsilon \to 0$, apparently turns out to be integrable (actually $N$ independent harmonic oscillators!), and the presence of the anharmonic term is responsible for the coupling of the normal modes, and the loss of integrability.
However, since we are interested in the behaviour when the system is very far from integrability, the use of the normal modes coordinates, although enlightening for the analytical estimates at low $\epsilon$, is of no help in the *strong stochasticity*

[15] By this expression we mean the purely kinematic effect of particle-particle correlation loss, entailed by the different *orbital times*, not to be confused with the *mixing* property of Ergodic theory. This explain also because very near to integrability, being the motion quasi periodic, such correlations persist in time, decreasing even the speed of this *phase mixing*, which is in turn responsible of the convergence of the curvatures to their asymptotic values. We already discussed the increasing effectiveness as the *thermodynamic limit* is approached. These remarks on the relevance of $\mathcal{N}$ are intended to warn about the risks hidden in extrapolating naively the outcomes of numerical simulations performed with too small $\mathcal{N}$; in particular for non extensive interaction potentials, like the gravity, where, as we will show, the $\mathcal{N}$-dependence is often hardly predictable. In these situations, the numerical evaluation of average values and dispersion of geometrodynamical quantities should be performed with long enough runs involving a suitably extended $\mathcal{N}$ range, to single out possible scaling behaviour.

[16] We will restrict to the physical case of three spatial dimensions, although many theoretical hints on the issue of relaxation in N-body systems have been achieved studying two or (mostly) one-dimensional systems, which display the very nice property to allow for computer simulations virtually free from numerical errors, [Tsuchiya et al. 1996].

regime, and it is superfluous from a numerical viewpoint, as it would imply a reduced efficiency of the algorithm.
To estimate the order of magnitude of various terms entering the Ricci curvature, we start inquiring on the *dynamical time scale* of the system. This is relevant to our analysis as it enters in the $s$ derivatives of the potential or kinetic energies. As most of interaction potentials, the FPU-$\beta$ model possess in the *genoma* its own lenght and time scales. These generally depend on some global parameters describing the macroscopic state of the system (as temperature and density). This DS is however peculiar in this respect being *isochronous* at low energy, as any harmonic system; i.e., when $\beta\epsilon \ll 1$, the global dynamical time-scale is of order unity. An important and well known property of this model, is the existence of a *strong stochasticity threshold* (SST) as the specific energy $\epsilon$ increases above a *critical value* $\epsilon_c$, being $\beta$ held fixed, distinguishing among two different *regimes* of Chaos, and also involving a transition on the features of the relaxation processes driving the system towards the equilibrium state, [Pettini and Cerruti-Sola 1991, Pettini 1993]. This SST is easily located by the intersection of two different scaling law of the maximal LCN, $\gamma_1$, with $\epsilon$: the first results, reported in the references above, seemed to support the claim that

$$\gamma_1(\epsilon) \propto \epsilon^2, \quad \text{for} \quad \epsilon \lesssim \epsilon_c \quad ; \qquad \gamma_1(\epsilon) \propto \epsilon^{2/3}, \quad \text{for} \quad \epsilon \gtrsim \epsilon_c \; ; \tag{50}$$

where $\epsilon_c \simeq \beta^{-1}$. A series of numerical computations with a large enough $N$ and carried out for very long integration intervals, [Cipriani 1993], extended the previously investigated range of specific energies, and while confirming the power law at low energies, find out that above the SST, the scaling is

$$\gamma_1(\epsilon) \propto \epsilon^{1/4}, \quad \text{for} \quad \epsilon \gtrsim \epsilon_c \; . \tag{51}$$

These outcomes are in complete agreement with the elegant SM canonical calculations, performed in the thermodynamic limit, in [Casetti and Pettini 1995, Casetti et al. 1995].
Within this context the GDA revealed is effectiveness in pointing out this threshold in a straightforward manner: figure 4 shows the behaviour or $\langle k_R(\epsilon)\rangle_s$ for different values of anharmonicity. The transition energy is neatly detected and it should be remarked the definitely quick convergence of the Ricci curvature to the average values there reported, for all values of $\beta\epsilon$, as it is shown in [Cipriani 1993, §5.3].
Moreover, within the GDA, it has become feasible to use a semi-analytic computation of the largest LCN, performed in [Cipriani 1993] using the computed time average, dispersion and correlation time of the Ricci curvature, and in [Casetti and Pettini 1995], using the same quantities, evaluated instead from a SM point of view, averaging over phase space.

The corrected scaling law behaviour found above the SST, though perhaps not crucial for what concerns the *existence* of two different regimes of Chaos, is relevant to our present discussion in that a simple calculation, [Cipriani 1993], shows that in the strongly anharmonic regime, the *dynamical time scale*, $t_D$, of the FPU-$\beta$ model scales exactly as $\epsilon^{-1/4}$, in such a way that the quantity $L_1 \stackrel{\text{def}}{=} \gamma_1 \cdot t_D$ is constant when strong Chaos has developed. This point is of outmost importance for the understanding of the nature of instability present in the gravitational N-body system.
In [Cipriani 1993], where it has been explored the dynamical behaviour of the FPU-$\beta$ system also for a range of values of the anarmonicity parameter, $\beta$, it has been shown that the relevant quantity that determines the level of stochasticity of this DS is, as expected, the product $\beta\epsilon$. Therefore, in the sequel, we will assume that the anharmonicity parameter is fixed (namely, $\beta = 0.1$) and let $\epsilon$ to vary. Moreover, all the quoted numerical results refer to simulations performed adopting periodic boundary conditions, i.e., $x_0 = x_{\mathcal{N}}, \; x_{\mathcal{N}+1} = x_1$.

Once established the scaling behaviour of the dynamical time with energy, we turn to the order of magnitude estimate of the quantities entering the Ricci curvature for the quartic FPU chain.
For example, the explicit computation of the $\mathcal{N}$-laplacian is straightforward and yields[17]

$$\frac{\Delta\mathcal{U}}{\mathcal{N}} = 2 + \frac{6\beta}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}(x_{i+1}-x_i)^2 = 2 + 6\beta\langle(\delta x)^2\rangle_{\mathcal{N}} \; , \tag{52}$$

which can be also written as (for the details, see [Cipriani 1993, Cipriani and Di Bari 1997b])

$$\frac{\Delta\mathcal{U}}{\mathcal{N}} \cong 2\left[1 + 3\left(\sqrt{1+\beta\epsilon} - 1\right)\right] \; ; \tag{53}$$

whereas for what concerns the dynamical time scale it is easy to find that

$$t_D = \left[\frac{\sqrt{1+4\beta\epsilon}-1}{2\beta\epsilon}\right]^{1/2} \mathcal{I}(c) \tag{54}$$

[17] Incidentally, this quantity also coincides with the Ricci curvature per degree of freedom in the Eisenhart metric, cfr. [Casetti and Pettini 1995].

where $\mathcal{I}(c)$ is almost constant dimensionless integral, and $c$ is a parameter measuring the departure from harmonic behaviour, and so is directly related to $\beta\epsilon$: $c = 1$ in the harmonic case, $c = 0$ in the purely *quartic* domain. It has to be remarked that $\mathcal{I}(c)$ is a monotonically increasing function of its argument, whose extremal values are

$$\mathcal{I}(0) = \frac{\sqrt{2}\,\pi^{3/2}}{4\left[\Gamma\left(\frac{3}{4}\right)\right]^2} \cong 1.311\ldots \quad ; \quad \mathcal{I}(1) = \frac{\pi}{2} \cong 1.5708\ldots$$

Similar expression can be found about the $(\mathcal{N}, \epsilon)$ dependence of all other quantities, as resumed in table 1. In [Cipriani and Di Bari 1997b], where a comparative analysis between FPU-$\beta$ model and self gravitating N-body system is also presented, we propose an explanation of the scaling law behaviours of Lyapunov exponents on the basis of these estimates, supplemented with analogous evaluations on the amplitude of fluctuations, *during* and *after* the *virialization* process.
A preliminary remark is devoted to the relation of quantities entering the frequency $Q(t)$ (or $\hat{k}_r(t)$) to the dynamical time $t_D$. While at intermediate $\epsilon$ values the relationship can't be singled out by eyes, in both the asymptotic regimes it is easy to see, that all the quantities behave as $t_D^{-2}$, the different weight they have into $Q(t)$ being solely due to their $\mathcal{N}$ dependences. We see from table 1 that among the *static* terms[18], the one related to the average squared force is always depressed by a factor $\mathcal{N}^{-a}$ with respect to the laplacian, with at least $a \gtrsim 1$. The use of the *central limit theorem* allows then to state that, for the FPU-$\beta$ chain, the *fluctuating* terms (see note 18), although comparable with the gradient term when the system is out of virial equilibrium, are then always negligible, in the large $\mathcal{N}$, with respect to $\Delta\mathcal{U}/\mathcal{N}$, and are even damped by a further factor of at least $\mathcal{O}(\mathcal{N}^{-1/2})$ or $\mathcal{O}(\mathcal{N}^{-1})$ when that equilibrium is attained. As the term for which the damping is less effective, i.e., that with the second order derivative with respect to time, has also a *zero time-average*, we understand why this it is important to fully appreciate the relevance of this transition in mdf systems.

On the grounds of the above discussion, we understand why the results obtained within the frameworks of the Jacobi and Eisenhart metrics agree for the FPU-$\beta$ model: the geodesic instability is driven by the fluctuations of the GDI's rather than by their negative values. For the FPU chain, all the terms which appear in $Q(t)$ in addition to $\Delta\mathcal{U}/\mathcal{N}$, which is nothing but the Ricci curvature per degree of freedom in the Eisenhart geometrodynamics, after the virialization phase has been completed can be neglected not only with respect to the average value of the latter, but even with respect to its own fluctuations, as the geodesics explore the manifold. This is generally true at all regimes, [Cipriani and Di Bari 1997b], and is always true above the SST, where the analytic computations of the maximal LCN obtained using some elementary results of the theory of Stochastic differential equations, [Van Kampen 1976], coincide, irrespective of the metric used. For small values of either $\beta\epsilon$ or $\mathcal{N}$, the quasi-periodic nature of the motions as well as the reduced effectiveness of the damping cause a much slower convergence to asymptotic values of the fluctuations, so reducing the reliability of finite time estimates.

## Long range, unscreened interactions: the gravitational N-body system.

When we focus our attention towards the gravitational N-body system, we are at once faced with an ambiguity which cannot be solved if considered only from a mathematical point of view. Along with physically relevant *real* peculiarity indeed, the Newtonian Gravity possess some mathematical singularities which do not actually involve any physical singular behaviour. So, in translating the dynamics into a geometric picture, we will try to cope with these mathematical divergencies without modifying the physically peculiar features of this interaction.

To describe the dynamics of $N$ bodies of mass $\{m_i\}$ interacting via the gravitational potential

$$V(r_{ij}) = -Gm_im_j/r_{ij} \ ,$$

we introduce the coordinates $\{q^a\}$ in the 3N-dimensional manifold $M_\mathcal{N}$

$$\mathcal{M} : \{q^a\} \ t.c. \ \mathcal{U}(\mathbf{q}) \leq E(< 0) \ ;$$

where:

$$q^a = \sqrt{m_i} r_i^\alpha \quad ; \quad a = 3(i-1) + \alpha \ \text{ and } \ i = 1, \ldots, N; \ \ \alpha = 1, 2, 3 \implies a = 1, \ldots, 3N = \mathcal{N}. \tag{55}$$

[18]This is only a rather generic terminology, to let it be understood that these terms do not involve explicit time derivatives. Of course, they also slowly evolve along with the process of approaching the global dynamical equilibrium, though in a very different fashion in the two DS's we are considering. Nevertheless, the *virial transition* is much less evident for them than for the other two terms, in such a way that, in the FPU-$\beta$ chain as well as in the gravitational N-body system too (although with a less clear hierarchy), after the first transients, the fluctuations responsible for the onset of Chaos come essentially from them alone.

endowed with the Jacobi metric, eq.(1), with the total potential energy given by

$$\mathcal{U}(\mathbf{q}) = \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} V(r_{ik}) \equiv \frac{1}{2} \sum_{i=1}^{N} \sum_{k \neq i} V(r_{ik}) \tag{56}$$

In order to get rid of unnecessary complications, we assume that all the particles have unit mass[19], in such a way that the total mass of the system is $N$, its average density $\bar{\rho} = N/V$, where $V$ is the volume occupied by the points. Amongst different possibilities, we selected this choice of units because it seems to be one of the most reasonable, and moreover allows for a direct interpretation of the results recently obtained, [Cerruti-Sola and Pettini 1995].

In analogy with what has been done for the FPU-$\beta$ model, we start to inquire about the possible existence of any scaling law of the dynamical time, $t_D$, for the N-body problem. This question is a very crucial one, because of the well known *scale-free* nature of Newtonian Gravity. Nevertheless, an N-body system do possess indeed a *natural time-scale*, which is known in stellar dynamics as the *orbital time*, because it represents a suitable average of the typical period of a body moving in the system. Neglecting some numerical factors of order unity, completely irrelevant for the present discussion, this dynamical time-scale is usually quoted in terms of the average mass density:

$$t_D \stackrel{\text{def}}{=} \frac{A}{\sqrt{\mathsf{G}\bar{\rho}}} \propto D^{3/2}/N^{1/2} \tag{57}$$

where $D \sim V^{1/3}$ represent the typical spatial extension of the system, and we have exploited our choice on the mass spectrum, neglecting again constant factors. It is easy to show that $t_D$ represents the right order of magnitude for most orbital periods even in a moderately inhomogeneus system, except for the small fraction of tightly bound *core objects*.

In a recent paper, [Cerruti-Sola and Pettini 1995], where for the first time were performed numerical simulations specifically devoted to support (or even correct) the GDA to Chaos for self-gravitating N-body systems, it has been reported, along with other interesting analyses, the absence of any threshold in the behaviour of the maximal Lyapunov exponent along with the increase of the energy density $\epsilon \stackrel{\text{def}}{=} E/N$. Moreover, it has been there discussed the seemingly non physical behaviour of this system when the number of degrees of freedom is varied. It has been found, indeed, that increasing the number of particles, the degree of stochasticity, as measured by the largest LCN[20], decreases, contrary to physical intuition, Statistical Mechanics expectations, and to what observed in any other mdf system considered before. The authors refused to believe in this apparent discrepancy, and attributed the cause to the too strong approximations which, starting from *exact* JLC equations of geodesic spread, led to an equation like our eq.(43). We now show that their numerical results are indeed correct, and that no failure can be attributed to the averaging process that lead firstly to equation (7) and down there to eqs.(28) and (43). The seemingly unphysical result, turns out to be instead completely reasonable, if we consider how the numerical simulations were performed[21]. As remarked above, all the bodies have been assigned unit mass, and initial conditions are generated from a spatially uniform distribution whose lenght scale $D$ is chosen to give the desired value of the total energy $E$, being the velocity (components) of the masses populated according to a gaussian distribution (in the CM coordinates system) whose variance, $\sigma_v^2$, is adjusted to fulfil the *virial theorem*, i.e., $2\,\langle \mathcal{T} \rangle = -\langle \mathcal{U} \rangle$, so that $D$ and $N$ completely determine the value of total energy, $E = \langle \mathcal{U} \rangle/2 \underset{N\to\infty}{\longrightarrow} \mathcal{U}/2$.

With this choice of initial configurations, in the case $N \gg 1$, the total energy ($G = m_i = 1$),

$$E \approx \frac{\mathcal{U}}{2} = -\frac{1}{2} \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} \frac{1}{r_{ik}}\ , \tag{58}$$

scales obviously as

$$E = -\frac{N(N-1)}{4} \langle r_{ik}^{-1} \rangle \cong -N(N-1)\frac{F_\rho}{D} \tag{59}$$

where $F_\rho$ is a factor of order unity depending on the actual density distribution, and it is so allowed to evolve slowly with time, along with the development of a typical *core-halo* structure, although this happens on time scales longer

[19] The problems we are now discussing do not depend on this point. It is obvious that this unrealistic assumption should be removed when issues as the problem of equipartition of energy, and the consequent *mass segregation* is addressed. But the conceptual question we are focusing on now would only be made heavier if a non uniform spectrum of masses is allowed.

[20] For a non compact phase space, as that of the newtonian N point masses, it is not rigorous to speak about *true* LCN's, but we can equally adopt this terminology, referring to the paper, [Cerruti-Sola and Pettini 1995] for a critical discussion of this item. Besides, we defer to the following discussion and to a forthcoming paper, [Cipriani and Di Bari 1997b], where some of the open questions will be addressed in a hopefully self consistent way.

[21] Actually these scaling laws, reported in table 1, do not depend qualitatively on this particular choice.

than those lasted usually by numerical simulations. So the total energy scales essentially as $|E| \approx N^2/D$. What this implies on the relationship between specific energy $\epsilon$ and the dynamical time scale $t_D$? The answer is straightforward: we have $|\epsilon| \approx N/D$; on the other end we have also

$$t_D = \frac{D^{3/2}}{N^{1/2}} \approx \frac{(N^2/|E|)^{3/2}}{N^{1/2}} = \frac{N}{|\epsilon|^{3/2}} \; ; \tag{60}$$

which is a result valid at any energy. Incidentally, we note that actually $t_D \sim D/\sigma_v$. This simple calculation explain all the results found in [Cerruti-Sola and Pettini 1995]; indeed the scaling behaviour of the largest LCN as $|\epsilon|^{3/2}$, with *N held fixed*, is consistent again, as for the FPU chain *above the* SST, with a constant value of the quantity $L_1 = \gamma \cdot t_D$. Even the (un)believed *unphysical* result of a decreased stochasticity when $N$ increases, turns out to be consequent to the neglecting of the $N$ dependence of $t_D$, and its interpretation becomes instead very clear within this picture: again, the dynamical time is proportional to $N$, if $\epsilon$ is held constant, in such a way that the instability exponent decreases in absolute value, but remains constant if measured in units of the dynamical frequency. In a sense, the lack of *stability* property, [Ruelle 1988], by the Newtonian potential, i.e., the *non-extensivity* of the potential energy, reflects itself on a dependence on $N$ of the dynamical time scale which do not occur in any customary interaction. The preceding discussion so contributes to give to the results presented in [Cerruti-Sola and Pettini 1995] a physical interpretation which largely confirms what argumented there and moreover support in a stronger way the use of the GDA, which turns out to be very fruitful even in the application to this peculiar and singular mdf system. It has been removed in fact the doubt about the reliability of the averaged equation (7), which really carries out almost all the informations about the qualitative behaviour of the dynamics of mdf Hamiltonian systems. The scaling law behaviours found numerically agree with a surprising precision with the analysis here proposed, for the LCN computation, and also for the evaluation of some quantities, named *Geometric Chaoticity Indicators* (*GCI*) by the authors: we suggest to look at figures 4 and 6 of the cited paper, to see how both the $\epsilon$ and $N$ dependences are perfectly fitted by the analytical estimates here presented (we remark that the scales used in the figures there are in natural logarithms).

Nevertheless, when we try to apply the GDA to gravitational N-body system, another issue should be necessarily addressed: whatever geometrization procedure is adopted, the term in the Ricci curvature (or even in the scalar one) which is usually *dominating* over the others has in this case a very singular behaviour. The laplacian of the Newtonian potential indeed vanishes everywhere except on the positions occupied by the sources of the field, where it diverges. As one of the most successful applications of the GDA for high dimensional DS's resides in the possibility of computation of *time and* SM *canonical averages* of geometrical quantities entering the evolution of perturbations, used to determine the stability of trajectories avoiding the explicit integration of the variational equations, it emerges clearly why we need a *recipe* to cope with such singularities. This problem is also related to the Statistical Mechanical properties of gravitational N-body systems that we will not touch here, [Cipriani and Di Bari 1997b]. If we perform a *static* average of the Ricci curvature over M, we neither need nor can to avoid these $\mathcal{O}(\mathcal{N}^2)$ singularities, which are nonetheless *Lebesgue-integrable*, and do not create any difficulty in the definition of the average values $\langle k_R \rangle_M$ or any other related quantity; although they contribute heavily to the magnitude of fluctuations around these averages. Vice versa, when we look at time averages, we face up to a problem of probability: what is the chance that a *true* collision among particles do occur? If we have to do with the purely mathematical problem, we are without hope: the GDI's possess their well defined static averages, to which the corresponding time averages never approach! We feel to be faced again, from a novel perspective, to the classical problem of the non existence of an equilibrium state for the *mathematical* Newtonian gravity, dressed within this framework as the *non existence* of an *ergodicity time*.

The way out to this problem is, as expected, a softening in the potential. This is an usual *trick* (either implicit or explicit) of N-body numerical simulators, adopted to cope with the so-called *close encounters* which occur sometimes during simulations. It is traditionally passed on that the occurrence of such kind of events is a consequence of the necessarily small number of particles used in a feasible simulation. This is true to a limited extent, from a numerical point of view, but it is a general occurrence, from a physical perspective, that during an orbital period some *few* stars suffer of such an event. We avoid to discuss here the details of the criteria adopted to define such a close encounter, [Cipriani and Di Bari 1997b], and limit ourselves to say that when the distance between any pair of *stars* becomes exceedingly small with respect to the average interparticle separation, $r_{ik} \ll d = D/N^{1/3}$, the newtonian interaction should break down, as other effects, possibly even non conservative, take place.
The easiest way to take into account for this breakdown consist to modify slightly (very very slightly) the interaction potential, allowing for a softening parameter, $\varpi$ (we don't use the *usual* $\varepsilon$ to avoid confusion with the specific energy $\epsilon$) enter in the *distance* between particles used to compute the potential:

$$r_{ik} \stackrel{\mathrm{def}}{=} \left[ (x_k - x_i)^2 + (y_k - y_i)^2 + (z_k - z_i)^2 + \varpi^2 \right]^{1/2} \tag{61}$$

The introduction of the softening, helps to improve the *conceptual* understanding of the peculiarities of Gravity without modifying appreciably the qualitative overall dynamics, which is determined by the long range nature of the interaction, rather than by the short range divergencies, involving only a negligible fraction of masses (when $N \gg 1$). The non-compact nature of the phase space, the *bad* statistical properties of gravitational interaction (that of being not a *stable* one) are indeed preserved by the introduction of the softening. This because it do not changes at all the long range behaviour of the interaction, neither introduces an *hard core*, as it could be erroneusly understood. The presence of the term $\varpi^2$ in the potential has so theoretical consequences by far more relevant than the practical ones obtained in the more well behaved numerical integration of equations of motion. Indeed, this *trick* allows both to evaluate analitically and to numerically compute, just for a check on those estimates, everywhere non-divergent quantities entering the GDI's, to find their scaling law behaviour with relevant global parameter of the system (i.e., the number of degrees of freedom $\mathcal{N} = 3N$ and the *energy density* $\epsilon$), by means of which to locate the possible relationships with the dynamical time-scale, $t_D$.

As we will see, the analytical estimates suffice to single out what is it the *dominant term* in the *frequency* $Q(t)$ appearing in equation (43), and also a precise determination of its relationship with the energy density $\epsilon$. For *standard* potentials, as the FPU-$\beta$ one, we have seen above that, provided $\mathcal{N} \gg 1$, neither the dynamical time $t_D$, nor the average values of $\hat{k}_R$ and $Q$ depends on $\mathcal{N}$, which weakly affects only the amplitude of the fluctuations. As expected, instead, the Gravity, ever corrected from too mathematical schematizations, shows a very different behaviour, disclosing a very intricate dependence of the GDI's on the number of particles $N$. This relationship can only be guessed through order of magnitude estimates, and we need to appeal to numerical simulations, whose results let us to state that:

- The frequency $Q(t)$ is overwhelmingly positive for any value of $N$, so the Chaos observed (or better, the exponential instability) is certainly due to parametric resonance.
- The Ricci curvature $k_R(s)$, and consequently $\hat{k}_R(t)$, is sign fluctuating for very small values of $N$, but shows a clear tendency to become more and more positive as the number of point masses increases. Already for some tens of bodies, it turns out to be mostly positive.
- In any case, the evidence of the scaling law with $N$ of all the terms appearing in $k_R(s)$, allows us to claim that, even for relatively small $N$, the average value of Ricci curvature is always positive, because the only term contributing with a constant negative sign is the faster to decrease with $N$.
- The evidence of a *virial transition* with the implied damping of fluctuations.
- If, moreover, the question of equivalence among static and dynamic averages is addressed, taking so into consideration the laplacian term, which is always the *biggest* one in the average, irrespective on the detailed way it is accounted for, the scaling law behaviour with $N$, prove that the Ricci curvature of the gravitational N-body system is almost everywhere (i.e., almost always) positive even for moderately large $N$. In the very limit $N \gg 1$, $k_R(s)$ is always rigorously positive.
- Well then, the geodesic instability too arise from the fluctuations of the *positive* Ricci curvature, which in turn implies the fluctuations (i.e., the anisotropy) of the sectional curvatures of the manifold. In such a case, neither rigorous results nor convincing arguments exist compelling the instability time-scale, of the order of the dynamical one, to give a guess on the SM relaxation time-scale.
- The absence of any transition, analogous to the SST observed in the FPU model, in the degree of chaoticity of the dynamics, leaves open the question of the existence of different regimes of rapidity in the approach to the equilibrium.
- The amplitude of fluctuations and the level of anisotropy in the $N$-body system are nevertheless amplified with respect to those occurring in the FPU chain. It is not hard to guess for the sources of the strong anisotropy and of the big fluctuations in the configuration manifold. The results obtained for few dimensional DS's, [Cipriani and Di Bari 1997c], let us to support the claim according to which a self gravitating $N$-body system find itself in a regime of fully developed chaos, for the vast majority of initial conditions, as long as $N \gg 1$.
- The discussion reported above is the starting point for an investigation on the issue of the existence of *one final* or *various partial* SM equilibria for $N$-body self gravitating systems. The analysis carried out on the relevance of the scaling law of the different terms appearing in the GDI's suggests a scenario with a full hierarchy of processes of approach to metastable equilibria, whose time scales are ordered in accordance to a criterion related to the scale (i.e., the fraction $n/N$ of the stars involved) of the process itself, and will appear soon.

## Comparison between analytical estimates and numerical results.

In the table 1 we report the analytical estimates, slightly improved with respect to those appearing in [Cipriani 1993], on the relevance and the $(\mathcal{N}, \epsilon)$ dependence of the terms entering the GDI's, whose full account will appear in [Cipriani and Di Bari 1997b], along with some consistency checks obtained by numerical simulations. Minor corrections, which do not change the overall behaviours, have been obtained taking into account a more careful statistical evaluation of the amplitude of those terms we named *fluctuating*. As it is evident from the table, for the FPU-$\beta$ chain there are no doubt on the relative weights of various terms. When the virialization phase is completed, and this happen at once, when the number of oscillators is appreciably greater than few tens[22], the average *positive* values of both $Q(t)$ and $\hat{k}_R(t)$ are determined mostly by the laplacian term, while the fluctuations around these values, again after the first phases of approach to virial equilibrium, are due exclusively to the laplacian for $Q(t)$, whereas are influenced also by other terms for what concerns $\hat{k}_R(t)$. Nevertheless, although characterized by greater fluctuations, the average values of $\hat{k}_R(t)$ obviously coincide with $\langle Q(t) \rangle$, as shown in fig.4. Moreover, as shown in fig.1, after the virial equilibrium has been attained, as long as we have $\mathcal{N} \gg 1$ and the system is far from integrability, the probability of the occurence of a negative value of the Ricci curvature becomes actually zero. Apart from this last remark, these results are substantially unaffected whether be the regime we are investigating, as all the terms scale with $\beta\epsilon$ in the same fashion, in such a way that their relative weights are unchanged. While, obviously, the amplitude of the fluctuations increases along with anharmonicity.
So, for the FPU-$\beta$ chain there aren't many cautions to be made, and the numerical simulations simply confirm the analytical estimations and support all the argumentations based on them, [Cipriani and Pucacco 1994a, Cipriani and Pucacco 1994b]. As a simple example we plot in figure 4, from [Cipriani 1993], the $\epsilon$ dependences of $\langle Q \rangle$ and $\langle \hat{k}_R \rangle$, for three different values of $\beta$, which exploits neatly the foreseen scaling as $(\beta\epsilon)^{1/2}$.

As remarked above, the situation for the $N$-body problem is more involved, and the analytical estimates alone let us know only some of the answers, although perhaps the most physically relevant ones, namely, where come from the most important contribution to $Q(t)$ *and* the scaling behaviours of the Ricci related quantities with the energy density $|\epsilon|$, which lead to the conclusion that all them depend on the energetic contents of the system exactly as they have to do; for example, that the average value, measured in *natural units*, $\langle Q \rangle \, t_D^2$ *do not depends on* $\epsilon$. Stated otherwise, this indicates that the scale invariance of gravity manifests itself within geometrical transcription in such a way that all the (possibly suitably rescaled to meet dimensional needs) averages of GDI's have the same values if measured in the unit of dynamical time scale $t_D$.
Moreover, having singled out the dominant contributions to $Q(t)$, we are able to definitely support our claims that even in this case is the *parametric resonance* to drive to the onset of Chaos, and that the system dynamics should undergo a transition along the approach to the virial, if the initial conditions are chosen out of equilibrium.
What it is left to determine is the scaling law behaviour of the GDI's with $N$, which is practically absent in *extensive* potentials, but nevertheless likely to exist in the *superextensive* N-body problem, as indeed shown in table 1. The correct behaviour has been established supplementing the analytical investigations with numerical simulations whose details will be presented elsewhere, but whose main outcomes are:

- The *frequency* $Q(t)$ is in almost always positive for the N-body problem, and the chances to find a negative value decrease quickly when $N$ increases. To show this, we report in figures 5, 6 and 7 the time evolution of all the contributions coming from *static* and *fluctuating* terms.
- Despite the positive values of both $Q(t)$ and $\hat{k}_R(t)$ (whose average is always positive and for which also the occurrence of local negative values becomes negligibly small when $N$ increases) imply that also the Ricci curvature per degree of freedom, $k_R(s)$, is positive; we found that the dynamics of self gravitating $N$-body systems is always strongly chaotic, [Cipriani and Di Bari 1997b], as far as chaoticity can be defined for non-compact phase spaces. This is explained again by the phenomenon of parametric resonance which is always in the regime of fully developed instability, as the fluctuations in the GDI's are for this DS constantly above the threshold required for the mechanism to onset.
- All quantities entering the GDI's have the same specific energy $\epsilon$ dependence as $t_D^{-2}$, i.e., both the laplacian and gradient related quantities, on one side, and the first (squared) and second time derivatives of the total potential

[22] We remind that the virial equations in their general form: $\langle q^a \partial H/\partial q^a \rangle = \langle p^a \partial H/\partial p^a \rangle$, where the averages are taken over time or phase space, when applied to mdf system, using the *central limit theorem* and summing over the degrees of freedom, let to state that the equality is true istantaneously with an approximation that becomes more and more reliable as $N$ increases, being the statistical *error* proportional to $N^{-b}$, where, depending on the interaction, $1 \lesssim b \lesssim 3$, see also note 15.

energy, on the other, scale exactly as $|\epsilon|^3$, in such a way that the *frequencies* they define are in constant ratio with the *natural* orbital one.

- The $N$ dependence of these quantities, even when removed that coming from the dynamical time (we recall that we have $t_D \propto N|\epsilon|^{-3/2}$), manifests a very peculiar behaviour, with all quantities changing with $N$, but in such a way that the hierarchy of their relative ratios either remains unchanged or becomes even more pronounced; figure 8 shows the behaviour of the *contributors* to the Ricci related frequencies when $N$ is varied.

- With the support of specifically devoted numerical simulations, we explained this peculiar $N$-dependence, which turns out to be very enlightening also on the consequences of the *gravitational clustering*, whose basic more elementary example is the formation of a binary system. The implications of this point on the evolutionary processes of stellar systems are currently under study.

- The numerical simulations performed confirmed that the analytical estimates correctly describe the behaviour with $N$ of the ratios between the terms containing the time-derivatives of the kinetic energy of the system.

- Also for the *static* terms the *a priori* estimates correctly indicate their relative weights and even their ratios to the *fluctuating* ones.

- We investigated in detail also the behaviour of the *static* terms, and interpreted their behaviour with $N$ on the basis of an analytical conjecture, confirmed by numerical computations, whose details will be discussed in [Cipriani and Di Bari 1997b], where we will show also how all these outcomes do not depend on the detailed way we adopted to cope with the mathematical singularity related to the laplacian term in order to restore its physically relevant role.

Figure 8 clearly shows the precise scaling law of all the quantities entering the GDI's on $N$, along with the trivial dependence on $|\epsilon|$, or, is is the same, on $t_D$. There are other minor comments which can be made to the results of numerical simulations, among which we mention the very small oscillations in the panels `a)` and `c)` of figure 2, along with their long overall *quasi-periodicity* (the time is in *millions* of harmonic units!). Moreover, panels `b)` and `d)` of the same figure point out how the relationship between Chaos and frequency of a negative sign of the Ricci curvature should be at least, *bimodal*.

## Conclusion.

This paper belongs to a line of research addressed to the issue of a *geometric description of Chaos* in DS's. Here we focused our attention mainly of mdf hamiltonian systems, in order to investigate the sources of instability and to single out possible *a priori* criteria to detect the onset of chaotic behaviour, in analogy with what obtained in the case of few dimensional DS's, [Cipriani and Di Bari 1997c].
We discussed how the criteria borrowed from abstract Ergodic theory could be used, if ever a realistic physical model would fulfil their rather restrictive hypotheses. To this goal we exhaustively investigated, starting at a very basic level, the relationships existing between the various *curvatures* that can be defined over a manifold M where the geodesic flow translating the dynamics in a geometrical picture takes place. We derived some of their logical consequences, and then indicated some cautious remarks in order to get reliable indications from ambitious applications of the GDA, [Gurzadyan and Savvidy 1986, Kandrup 1990].
In particular, we here reported the formal explanation (see,`e.g.`, [Cipriani 1993]), within a more general context, of the analytical and numerical results we obtained, [Cipriani and Pucacco 1994a, Di Bari and Cipriani 1997c, Cipriani and Di Bari 1997c], about the irrelevance of scalar curvature and related quantities to detect the presence of instability, obviously as long as $\mathcal{N} \geq 3$.
It is only recently, `e.g.`, [Pettini 1993, Cipriani 1993], that a careful framework has started to be set up. Thanks to this more systematic approach, the method has revealed, and we claim only in part, its fruitfulness, for both mdf and few dimensional DS's.
In the present work we exploited its ability to single out the transition between two different regimes of Chaos in the FPU-$\beta$ model. Further, we also reinterpreted some sources of misunderstandings for what concerns the study of instability in the gravitational N-body problem. We shown its reliability performing analytical calculations, aided and supported by numerical computations, able to detect the scaling law behaviour of relevant quantities entering the equations governing the second order variational equations and in such a way identifying once more the peculiar

behaviour on Newtonian Gravity. We proved nevertheless that the mechanisms responsible of the onset of Chaos in *standard* mdf Hamitonian systems of interest in a classical Statistical Mechanics context, are also at work to determine the *exponential instability* (if not *true Chaos*), in this *not well behaved* DS. In this way, the somewhere persistent claims, [Gurzadyan and Savvidy 1986, Kandrup 1990], about a naïve relationship between instability and *relaxation* time scales should be revisited, and that their identification should be considered definitely false. Rather, it has to be reminded that for some DS's there are some analogies in their behaviours, though limited to the existence of a *common transition* between two different regimes, [Pettini and Cerruti-Sola 1991], but that, for the N-body system no relationship has any more than heuristic support, [Cipriani and Pucacco 1994a, Cipriani and Pucacco 1994b].
The GDA has made possible, the *analytical evaluation of the maximal* LCN for the FPU chain, through simple procedures developed in the framework of stochastic differential equations, [Van Kampen 1976],using solely the time average of the Ricci curvature and of its dispersion along with an estimate of its autocorrelation time (see [Cipriani 1993, Cipriani and Di Bari 1997b] and [Casetti and Pettini 1995]). The analytically calculated maximal LCN's are in a definitely satisfactory quantitative agreement with those numerically computed using the standard tangent dynamics equations, except at a very low (specific) energy, i.e., when Chaos is not yet fully developed, as in this case *the fluctuations* of GDI's show a slow convergence to their asymptotic values (whereas the average values of the curvatures are reached very quickly also in the harmonic regime). Incidentally, this is a signature of the many faces of the approach to *the equilibrium*, which is a concept strongly dependent on the observable you are looking to *relax* to its *ergodic value*. Recalling a previous discussion, we furthermore observe that similar results are obtained averaging over phase space, instead of on time, and it should be intriguing to ask how the GDA allows also to gain some hints on the relaxation time scales for mdf systems whose final equilibrium is guaranteed to exist.
The framework outlined here and previously developed in [Cipriani 1993, Di Bari 1996] serves as a starting point to try to answer also these basic questions. In the near future we will exploit the results obtained both in the case of few dimensional DS's and for mdf lagrangian systems, to shed light, from a novel perspective, to the long standing issue of a dynamical justification of the statistical approach in (classical) mechanics.

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# References


[Anosov 1967] Anosov D. V. (1967) *Proc.Steklov Inst. Math.* **90**, 1.

[Arnold 1980] Arnold V.I. (1980) *Mathematical Methods of Classical Mechanics* Springer-Verlag.

[Benettin et al. 1977] Benettin G., Brambilla R. and Galgani L. (1977) Physica, **87A**, 381.

[Benettin 1988] Benettin G. (1988) in *Non linear evolution and Chaotic phenomena*, (G.Gallavotti - P.F. Zweifel eds.), Plenum, 121.

[Benettin et al. 1984] Benettin G., Galgani L. and Giorgilli A. (1984) *Nature*, **311**, 444.

[Berndt and Vanhecke 1992] Berndt J. and Vanhecke L. (1992) *Differ. Geom. and Appl.*, **2**, 57.

[Casati 1975] Casati G. (1975) Lett. N. Cimento, **14**, n.9, 311.

[Casetti and Pettini 1995] Casetti L. and Pettini M. (1993) *Phys. Rev. E*, **48**, 4320.

[Casetti et al. 1995] Casetti L., Livi R. and Pettini M. (1995) *Phys. Rev. Lett.*, **74**, 375.

[Cerruti-Sola and Pettini 1995] Cerruti-Sola M. and Pettini M. (1995) *Phys. Rev. E*, **51**, 53.

[Cerruti-Sola and Pettini 1996] Cerruti-Sola M. and Pettini M. (1996) *Phys Rev. E*, **53**, 179.

[Cipriani 1993] Cipriani P. (1993) *Ph.D.Thesis*, Univ. of Rome "La Sapienza", (in italian).

[Cipriani and Pucacco 1994a] Cipriani P. and Pucacco G. (1994) *N. Cimento B*, **109**, 325.

[Cipriani and Pucacco 1994b] Cipriani P. and Pucacco G. (1994), in *Ergodic Concepts in Stellar Dynamics*, (V.G. Gurzadyan - D. Pfenniger eds.), Lecture Notes in Physics:vol.430, Springer-Verlag, 163.

[Cipriani et al. 1996] Cipriani P., Pucacco G., Boccaletti D. and Di Bari M. (1996) in *Chaos in Gravitational N-body System*, (eds. J.C.Muzzio et al.), Kluwer, 167.

[Cipriani and Di Bari 1997a] Cipriani P. and Di Bari M. (1997), in *Dynamics of small bodies in the Solar System*, (A.E. Roy - B. Steves eds.), Plenum, xxx.

[Cipriani and Di Bari 1997b] Cipriani P. and Di Bari M. (1997), submitted.

[Cipriani and Di Bari 1997c] Cipriani P. and Di Bari M. (1997), submitted.

[Di Bari 1996] Di Bari M. (1996) *Ph.D.Thesis*, Univ. of Rome "La Sapienza" (in italian).

[Di Bari et al. 1997] Di Bari M., Boccaletti D., Cipriani P. and Pucacco G. (1997) *Phys. Rev. E*, **55**, 6448.

[Di Bari and Cipriani 1997a] Di Bari M. and Cipriani P. (1997) in *Proc. 12th Italian Conference on General Relativity and Gravitational Physics*, World Scientific, xxx.

[Di Bari and Cipriani 1997b] Di Bari M. and Cipriani P. (1997) *Planetary and Space Science this number*, xxx.

[Di Bari and Cipriani 1997c] Di Bari M. and Cipriani P. (1997) submitted.

[Eisenhart 1929] Eisenhart L.P. (1929) Ann. of Math. **30**, 591.

[Farquhar 1964] Farquhar I.E. (1964) *Ergodic Theory in Statistical Mechanics.*, Interscience Publishers.

[Fermi et al. 1965] Fermi E., Pasta J. and Ulam S. (1965) in *Collected Papers of E. Fermi* **vol.II**, (E.Segré ed.), Univ. of Chicago Press, 978.

[Galgani 1988] Galgani L. (1988) in *Non linear evolution and Chaotic phenomena*, (G.Gallavotti - P.F. Zweifel eds.), Plenum, 147.

[Gallavotti 1995] Gallavotti G. (1995) *Meccanica Statistica*, G.N.F.M.-C.N.R., (in italian).

[Goldstein 1980] Goldstein H. (1980) *Classical Mechanics* ($2^{nd}$ edition), Addison-Wesley.

[Gurzadyan and Savvidy 1986] Gurzadyan V.G. and Savvidy G.K. (1986) *Astron.Astroph.* **160**, 203.

[Kandrup 1990] Kandrup H.E. (1990) *Ap.J.*, **364**, 420.

[Katok and Hasselblatt 1995] Katok A. and Hasselblatt B. (1995) *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press.

[Krylov 1979] Krylov N.S. (1979) *Works on Foundations on Statistical Physics*, Princeton Univ. Press.

[Landau and Lifsits 1980] Landau L.D. and Lifsits E.M. (1976) *Mechanics* ($3^{rd}$ edition), Pergamon.

[Ma 1988] Ma K.S. (1988) *Statistical Mechanics*, World Scientific.

[Morbidelli and Froeschlé 1996] Morbidelli A. and Froeschlé C. (1996) *Celest. Mech. and Dynam. Astron.*, **65**, 227.

[Pettini and Cerruti-Sola 1991] M.Pettini M. and Cerruti-Sola M. (1991) *Phys. Rev. A*, **44**, 975.

[Pettini 1993] Pettini M. (1993) *Phys. Rev. E*, **47**, 828.

[Rund 1959] Rund H. (1959) *The differential Geometry of Finsler Spaces*, Springer-Verlag.

[Ruelle 1988] Ruelle D. (1988) *Statistical Mechanics: Rigorous Results* Addison-Wesley.

[Sinai 1991] Sinai Ya.G. (1991) (Editor) *Dynamical Systems*, World Scientific.

[Synge 1926] Synge J.L. (1926) *Phil. Trans. A* **226** 31.

[Synge and Schild 1978] Synge J.L. and Schild A. (1978) *Tensor Calculus* Dover (reprint of original edition 1949).

[Toda 1974] Toda M. (1974) Phys. Lett, **48A**, 335.

[Toda et al. 1992] Toda M., Kubo R. and Saitô N. (1992) *Statistical Physics I*, Springer-Verlag.

[Tsuchiya et al. 1996] Tsuchiya T., Gouda N. and Konishi T. (1996) *Phys. Rev. E* **53**, 2210.

[Szydlowski and Krawiec 1993] Szydlowski M. and Krawiec A. (1993) *Phys. Rev. D*, **47**, 5323.

[Van Kampen 1976] Van Kampen N.G. (1976) *Phys. Rep.*, **24**, 171.

# Figure Captions

Figure 1: Long time behaviour of the measured *cumulative frequency* of occurrence of negative values of the Ricci curvature $F_{-}(t)$ in the FPU-$\beta$ chain, for a set of initial conditions *far enough* from integrability. All the solid lines refer to $N = 450$ coupled oscillators with $\beta\epsilon$ ranging from $5 \cdot 10^{-3}$ to $10^3$. The dashed lines represent the behaviour of a smaller system ($N = 150$) not too far from integrability, $\beta\epsilon = 10^{-3},\ 10^{-2}$, from bottom to top, showing the greater noise affecting simulations with too small $N$.

Figure 2: The same of the figure 1, but in the opposite situation: very small $N$ and/or conditions very near to integrability. The values of $(N, \beta\epsilon)$ are: a) $(50, 5 \cdot 10^{-5})$; b) $(50, 0.03)$ (upper curve) and $(50, 0.01)$ (lower curve); c) $(50, 5 \cdot 10^{-4})$; d), from top to bottom, $(150, 5 \cdot 10^{-5})$, $(150, 10^{-4})$, $(450, 5 \cdot 10^{-5})$ and $(150, 2 \cdot 10^{-4})$. The horizontal scale now is always linear and time is there expressed in million of units.

Figure 3: Energy dependence of the largest LCN of the FPU-$\beta$ chain, computed in analytical way. The dashed lines trace the low and high energy slopes: $(\beta\epsilon)^2$ and $(\beta\epsilon)^{1/4}$, respectively. In the small panel we plotted the *asymptotic* values of the cumulative frequency $F_{-}(t)$ for the same conditions used to compute the LCN's.

Figure 4: Time averages of the *rescaled* Ricci curvature $\hat{k}_R(t)$ and of the *Hill frequency* $Q(t)$ for the same set of initial conditions use to compute the largest LCN. The continuous line is the plot of the analytic estimate reported in table 1. The SST is neatly visible.

Figure 5: Time behaviour of the quantitities entering the GDI's, $Q(t)$ or $\hat{k}_R(t)$, for the gravitational N-body system with $N = 10$ (!) and $|\epsilon| = 1$. The small panels show, from left to right, the contributions to $Q(t)$ (shown alone in the larger window) from the *fluctuating* and *static* terms, respectively. In the small panels here and in the following figures, the laplacian term is indicated with a solid line, the squared gradient as dot-dashed, the (positive values) of the second derivative of the potential energy as dashed, and the dotted curve display its squared first time derivative. In this and the following figures all the terms are plotted with the weights they enter in the expression of $Q(t)$.

Figure 6: The same as figure 5, but for $N = 100$. Note the logaritmic vertical scale of the inserted plot. The quantities are represented as in the previous figure.

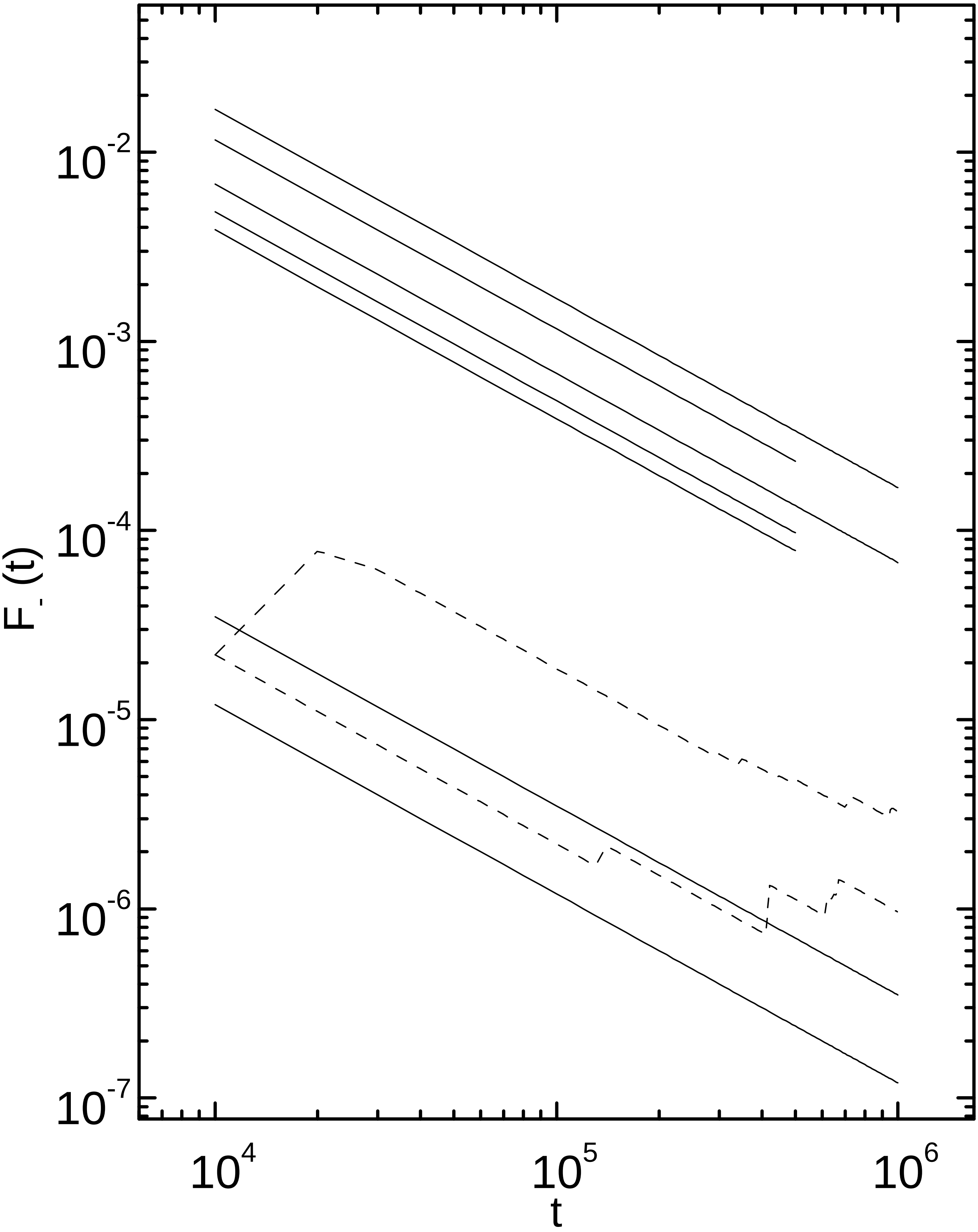
Figure 7: The same as the previous two figures but for $N = 200$ and with all the quantities in a plot. The small solid triangles trace the behaviour of $Q(t)$ in order to try to disentangle it from that of $\Delta\mathcal{U}/\mathcal{N}$, with which practically coincides. The other lines confirm the relative hierarchy of other terms.

Figure 8: In the large area we plot the $N$ dependence of the time averages of the various terms. These different scaling with $N$ are explictly shown in the inserted plot, which also points out the *sharp* $|\epsilon|^3$-dependence of all these quantities.

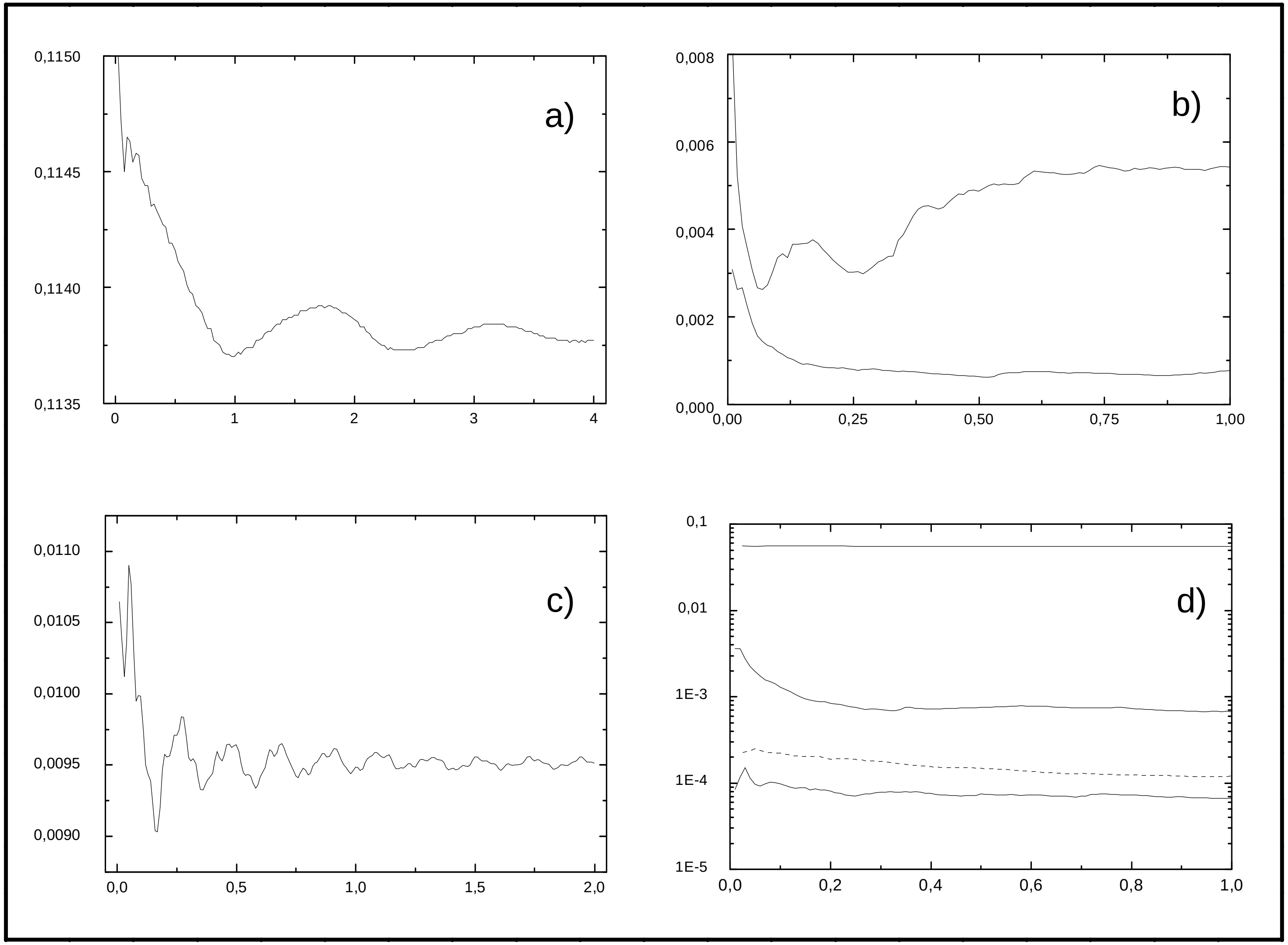
# Tables

Table 1: Analytical estimates of the $(N,\epsilon)$-dependence of quantities entering the GDI's and of the *dynamical* time, for the FPU-$\beta$ chain and the Gravitational N-body system, along with the indication about the scaling law behaviours *observed* for the instability exponents of both dynamical systems, evaluated either numerically or using the semi-analytical method discussed in the text. In this table $N$ represents the number of oscillators or the number of *masses*. We remark that the amplitudes here indicated should be taken as *order of magnitude estimates*, up to numerical factors of order unity. Moreover, for what concerns the *fluctuating terms*, these evaluations represent almost the largest absolute value, as they average can be much smaller (even vanishing!). The quantities indicated with the symbols $C_{F_n}$ depend on the clustering level of the N-body system and then slowly evolve, $\eta$ is a tracer of the way we adopted to cope with the newtonian singularity and the exponents $a_n$ reflect the efficiency of the *virial damping* on the amplitudes of evolving terms. Their explicit expression, as well that of the analytical formula, $\Lambda$, to compute the LCN, is given in [Cipriani 1993, Cipriani and Di Bari 1997b], see also [Casetti and Pettini 1995].

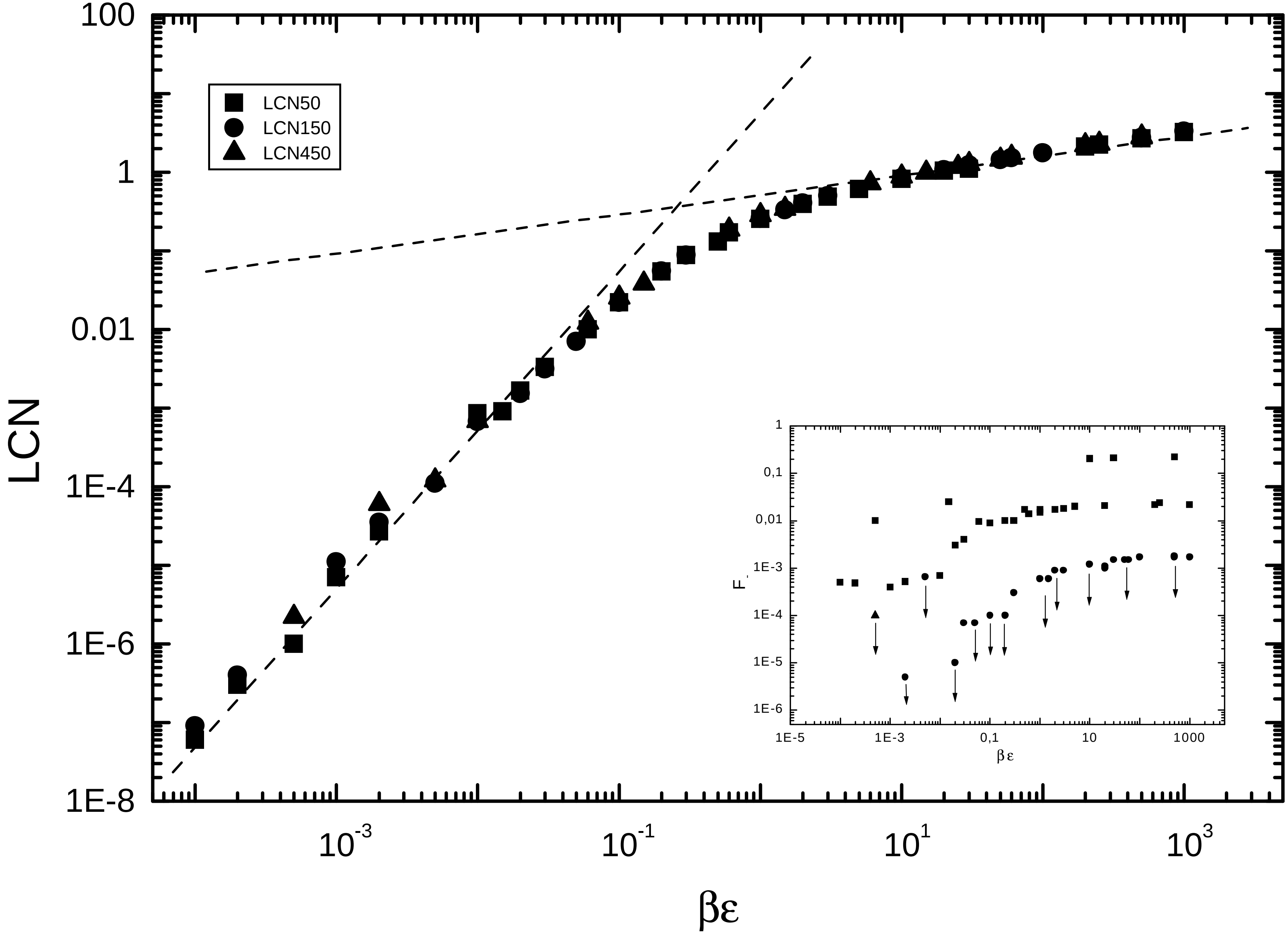
| | **FPU** | | | **N-Body** | |
|---|---|---|---|---|---|
| Quantity | Amplitude | Limiting Behaviour $\beta\epsilon \ll 1$ <br> $\beta\epsilon \gg 1$ | VDF | Amplitude | VDF |
| $\frac{\Delta\mathcal{U}}{N}$ | $2+6\left(\sqrt{1+\beta\epsilon}-1\right)$ | $2+3\beta\epsilon+\mathcal{O}(\beta\epsilon)^2$ <br> $6\sqrt{\beta\epsilon}\ +\mathcal{O}(\beta\epsilon)^{-1/2}$ | $\mathcal{O}(1)$ | $\eta^2\frac{\lvert\epsilon\rvert^3}{N^2}C_{F_5}$ | $\mathcal{O}(1)$ |
| $\frac{(\mathrm{grad}\,\mathcal{U})^2}{N\mathcal{W}}$ | $(1+\beta\epsilon)\frac{\sqrt{1+\beta\epsilon}-1}{N\beta\epsilon}$ | $(1+\beta\epsilon)\ /\ 2N$ <br> $\sqrt{\beta\epsilon}/N$ | $\mathcal{O}(1)$ | $\frac{\lvert\epsilon\rvert^3}{N^4}C_{F_4}$ | $\mathcal{O}(N^{a_3})$ |
| $\frac{\lvert\ddot{\mathcal{U}}\rvert}{N\mathcal{W}}$ | $(1+\beta\epsilon)\frac{\sqrt{1+\beta\epsilon}-1}{N^{3/2}\beta\epsilon}$ | $\sim N^{-1/2}\frac{(\mathrm{grad}\,\mathcal{U})^2}{N\mathcal{W}}$ | $\mathcal{O}(N^{a_1})$ | $\frac{\lvert\epsilon\rvert^3}{N^4}C_{F_4}$ | $\mathcal{O}(N^{a_2})$ |
| $\frac{1}{N}\left(\frac{\dot{\mathcal{U}}}{\mathcal{W}}\right)^2$ | $(1+\beta\epsilon)\frac{\sqrt{1+\beta\epsilon}-1}{N^2\beta\epsilon}$ | $\sim N^{-1}\frac{(\mathrm{grad}\,\mathcal{U})^2}{N\mathcal{W}}$ | $\mathcal{O}(N^{2a_1})$ | $\frac{\lvert\epsilon\rvert^3}{N^5}C_{F_4}$ | $\mathcal{O}(N^{2a_2})$ |
| $t_D$ | $\left(\frac{\sqrt{1+4\beta\epsilon}-1}{2\beta\epsilon}\right)^{1/2}$ | $1-\beta\epsilon/2+\mathcal{O}(\beta\epsilon)^2$ <br> $(\beta\epsilon)^{-1/4}\ +\mathcal{O}(\beta\epsilon)^{-3/4}$ | – | $\frac{N}{\lvert\epsilon\rvert^{3/2}}$ | – |
| $LCN$ | $\Lambda\left(\bar{\chi},\sigma_\chi,\tau_\chi\right)$ | $(\beta\epsilon)^2$ <br> $(\beta\epsilon)^{1/4}$ | – | $\frac{\lvert\epsilon\rvert^{3/2}}{N}$ | – |



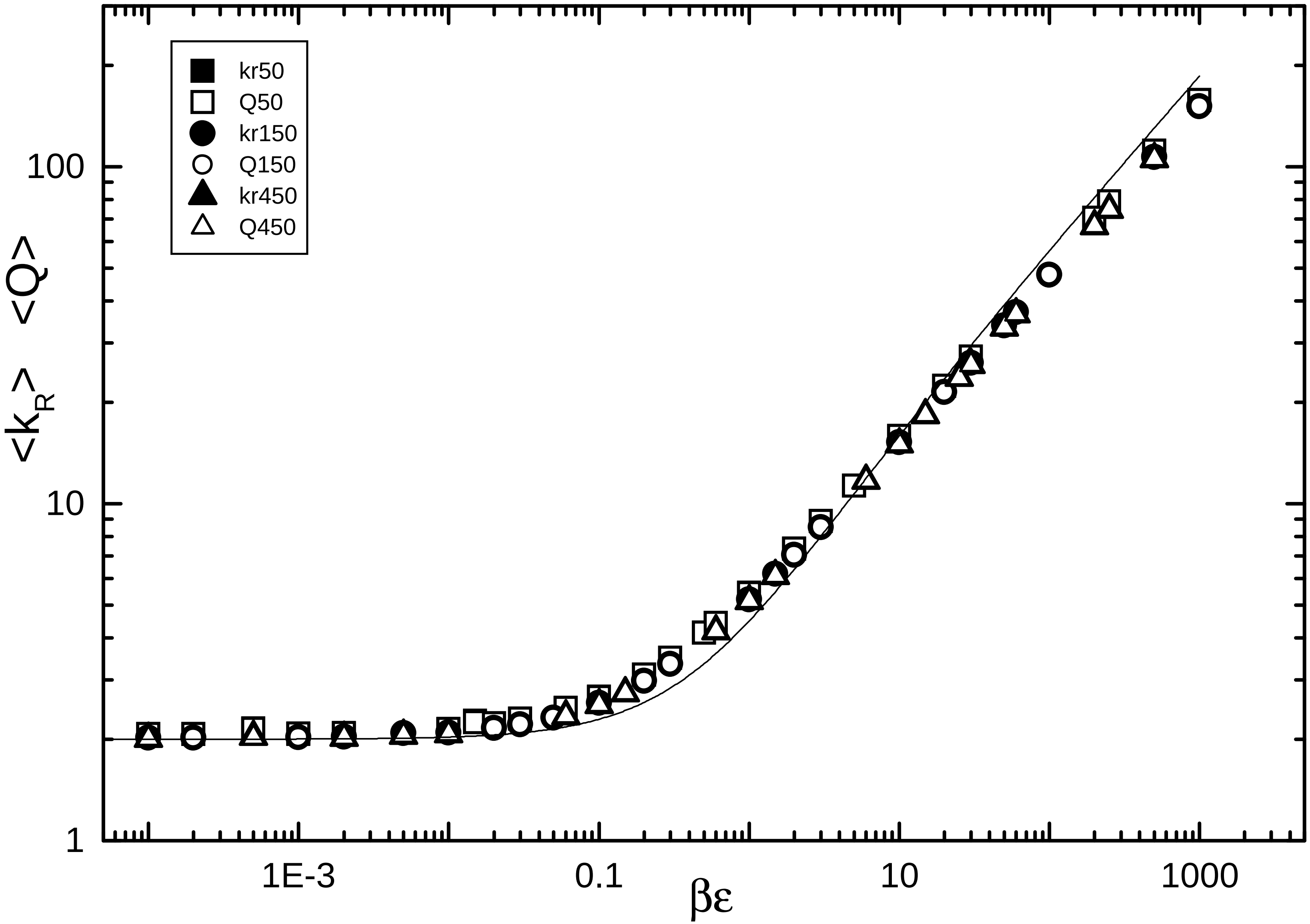
$10^{-2}$
$10^{-3}$
$10^{-4}$
$10^{-5}$
$10^{-6}$
$10^{-7}$
$F_-(t)$
$10^4$
$10^5$
$10^6$
t



0,1150
0,1145
0,1140
0,1135
0
1
2
3
4
a)
0,008
0,006
0,004
0,002
0,000
0,00
0,25
0,50
0,75
1,00
b)
0,0110
0,0105
0,0100
0,0095
0,0090
0,0
0,5
1,0
1,5
2,0
c)
0,1
0,01
1E-3
1E-4
1E-5
0,0
0,2
0,4
0,6
0,8
1,0
d)



LCN
βε
100
1
0.01
1E-4
1E-6
1E-8
$10^{-3}$
$10^{-1}$
$10^{1}$
$10^{3}$
LCN50
LCN150
LCN450
F
1
0,1
0,01
1E-3
1E-4
1E-5
1E-6
1E-5
1E-3
0,1
10
1000
β ε



kr50
Q50
kr150
Q150
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Q450
100
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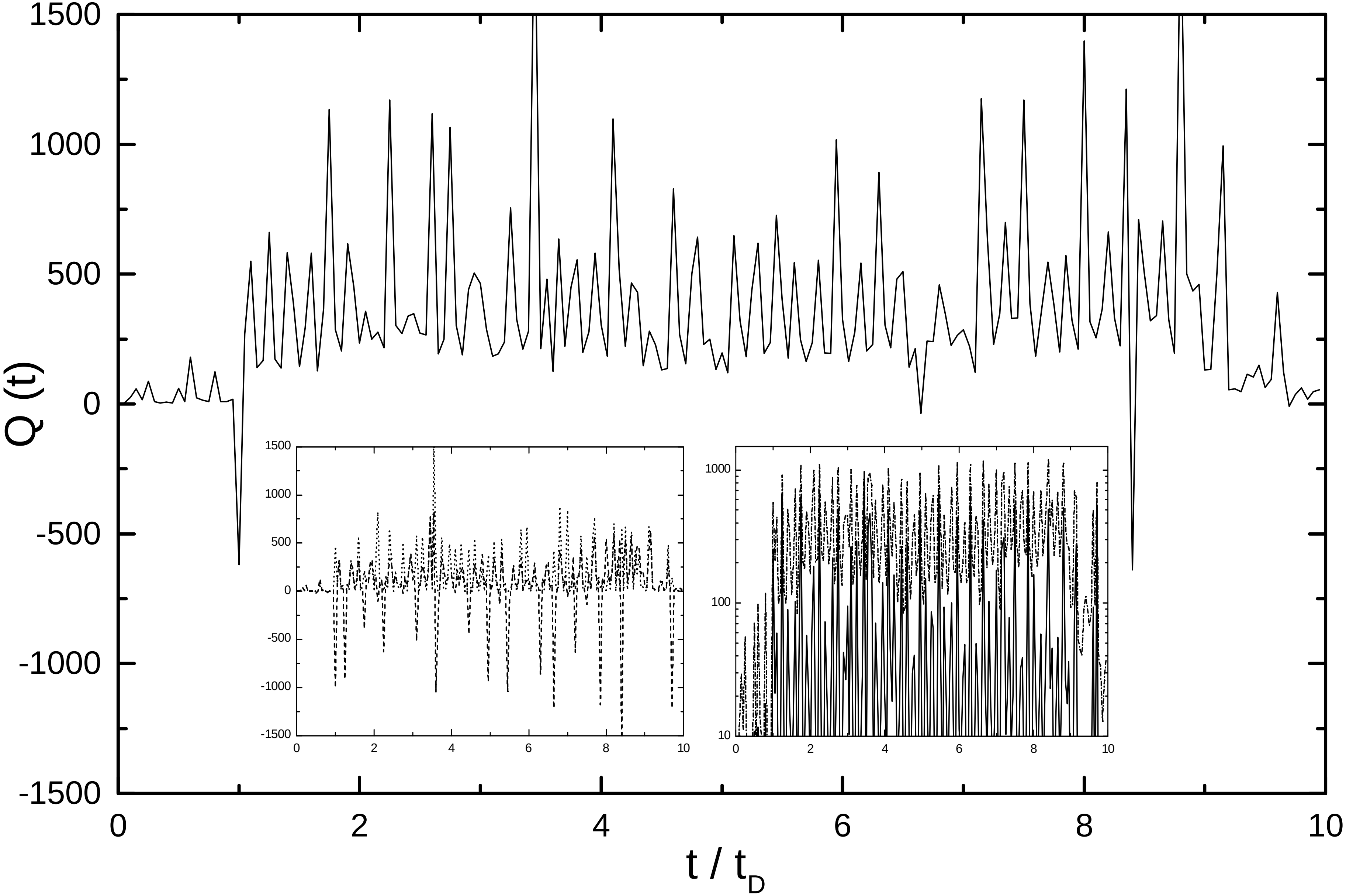
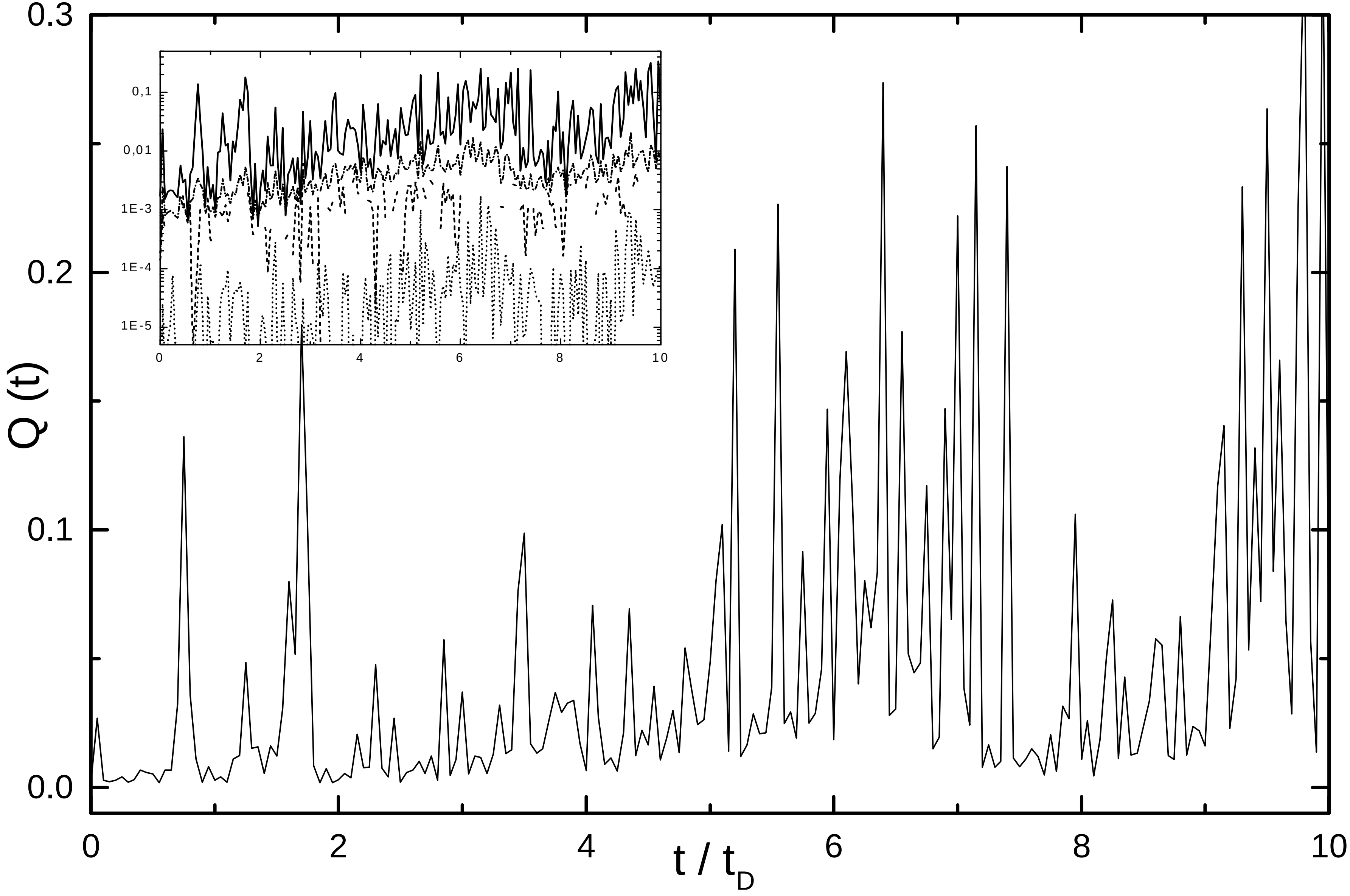
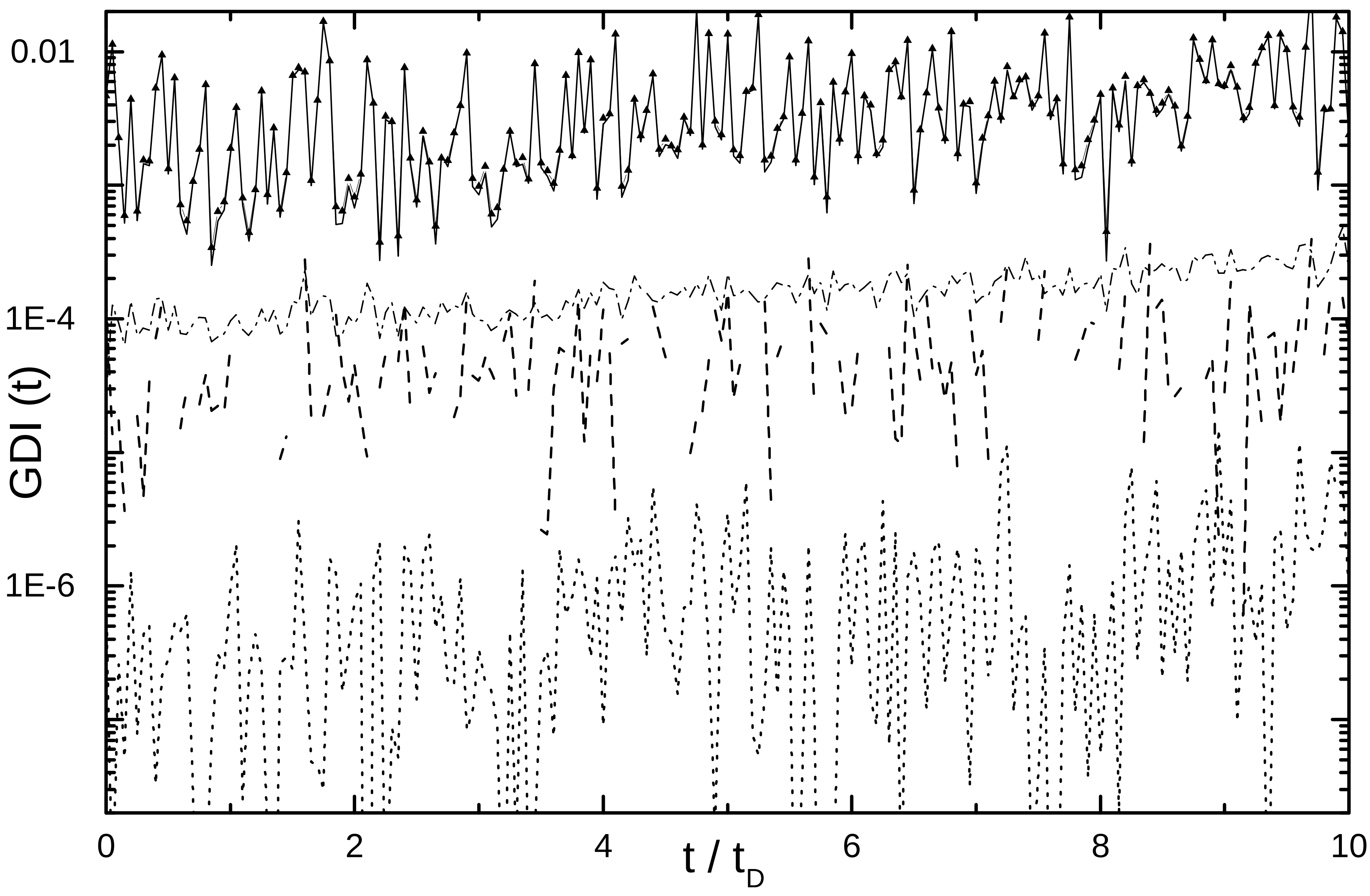
10
1000
βε
<k_R> <Q>

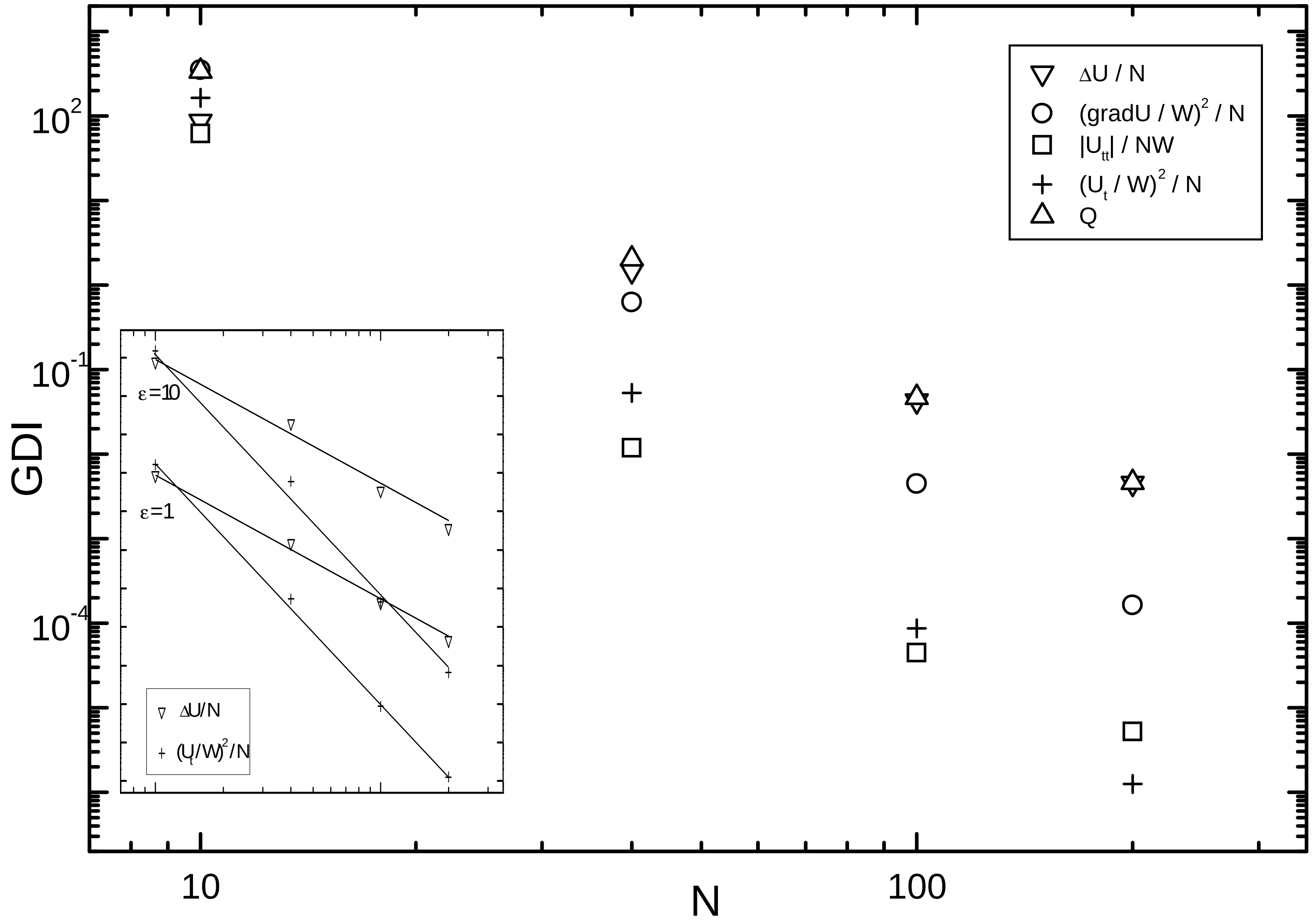


Q (t)
t / $t_D$
1500
1000
500
0
-500
-1000
-1500
0
2
4
6
8
10


0.3
0.2
0.1
0.0
Q (t)
0
2
4
6
8
10
t / $t_D$
0,1
0,01
1E-3
1E-4
1E-5
0
2
4
6
8
10



0.01
1E-4
1E-6
GDI (t)
0
2
4
6
8
10
t / t_D



GDI
N
10
100
$10^2$
$10^{-1}$
$10^{-4}$
$\Delta U / N$
$(\mathrm{grad}U / W)^2 / N$
$|U_{tt}| / NW$
$(U_t / W)^2 / N$
Q
$\varepsilon=10$
$\varepsilon=1$
$\Delta U/N$
$(U_t/W)^2/N$